A dual certificate analysis of compressive off-the-grid recovery

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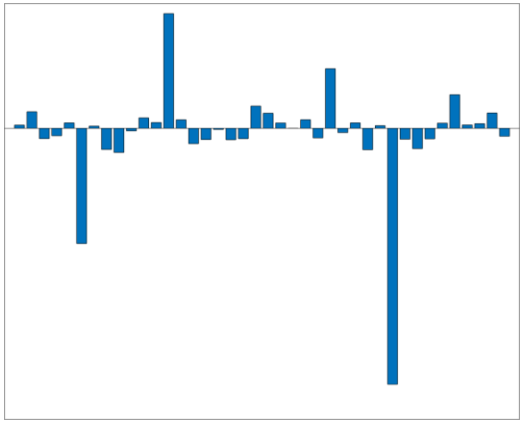
Joint work with **Clarice Poon** (Cambridge Uni.), **Gabriel Peyré** (ENS)

Journée GdR MIA, May 3rd 2018

Discrete compressive sensing



$$x \in \mathbb{R}^n$$

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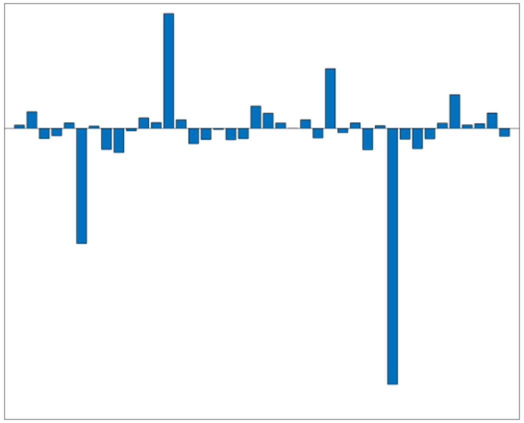

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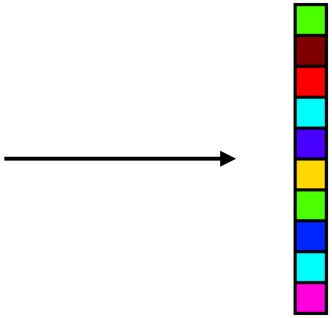
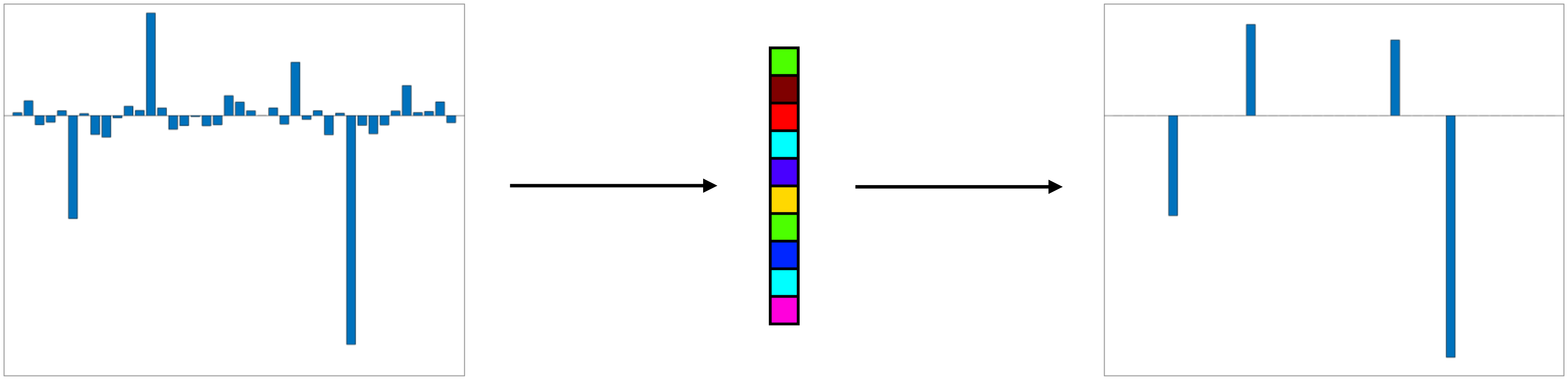


$y = Mx + e$

Discrete compressive sensing



$x \in \mathbb{R}^n$ $\quad$ $y = Mx + e$ $\quad$ $\tilde{x}$

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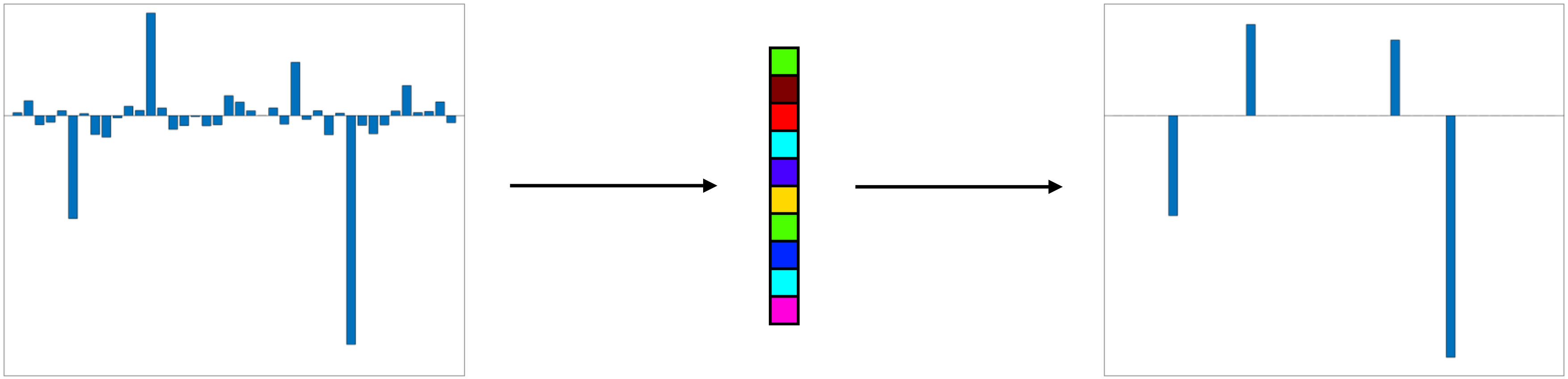

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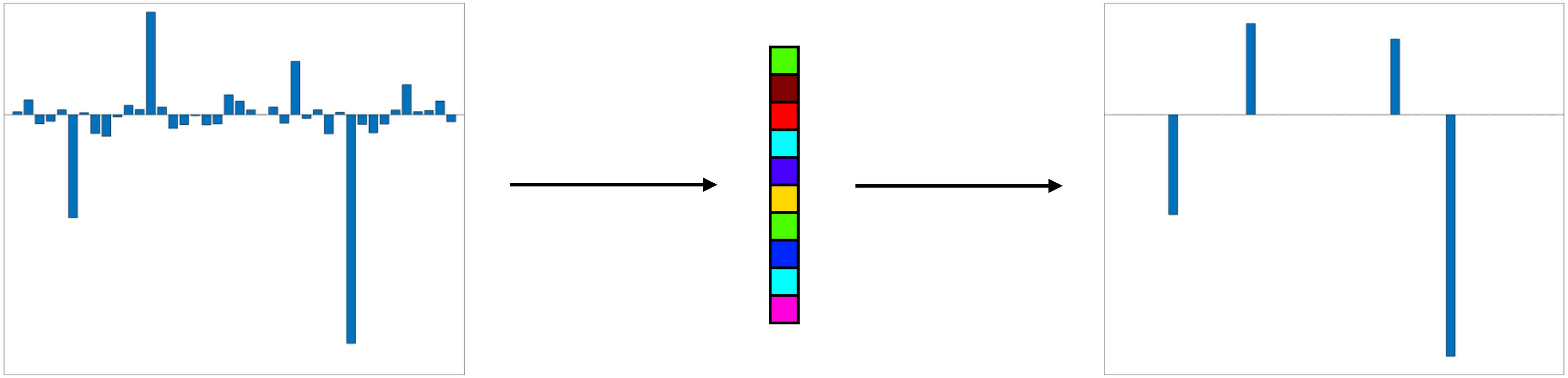
- **Signal**: vector

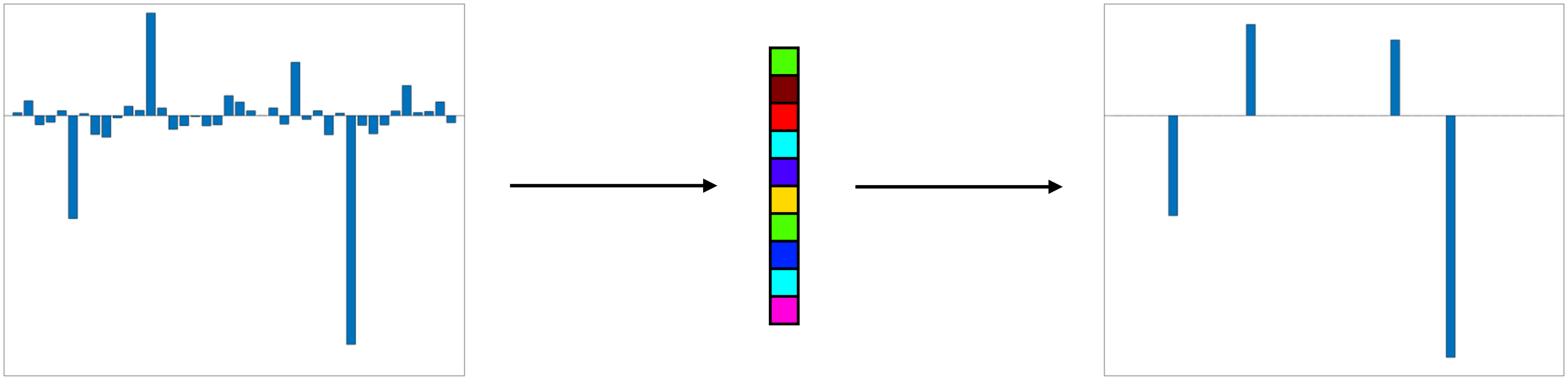
Discrete compressive sensing



$x \in \mathbb{R}^n$ $\quad$ $y = Mx + e$ $\quad$ $\tilde{x}$

- **Signal**: vector
- **Sparsity**: few non-zeros coefficients

Discrete compressive sensing



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- **Dimensionality reduction** (often random matrix)

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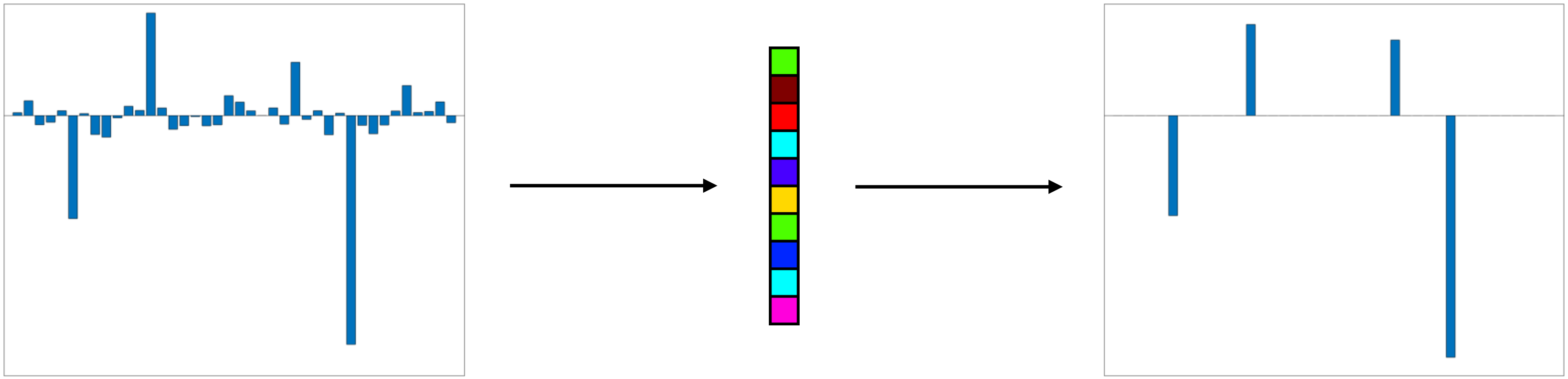

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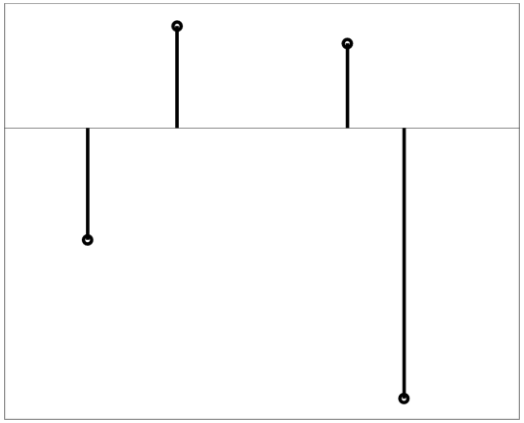
- **Signal**: vector
- **Sparsity**: few non-zeros coefficients
- **Dimensionality reduction** (often random matrix)
- **Recovery**: convex relaxation

$$\min_{\|x\|_0 \leq s} \|Mx - y\| \longrightarrow$$

LASSO

$$\min_x \frac{1}{2}\|Mx - y\|_2^2 + \lambda\|x\|_1$$

Off-the-grid recovery: « super-resolution »



$$\mu \in \mathcal{M}(\mathcal{X})$$

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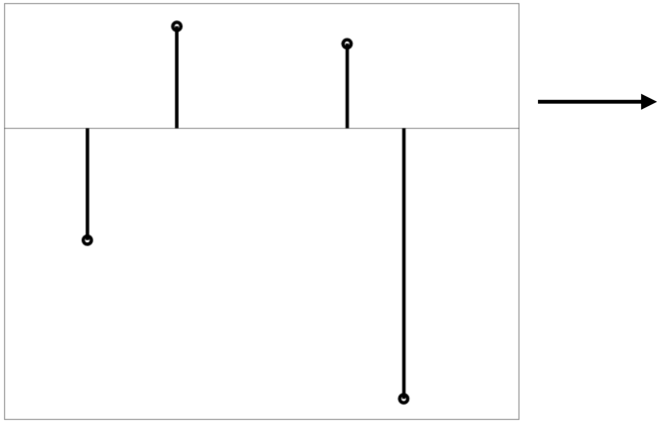

$$y = \Phi\mu + e$$

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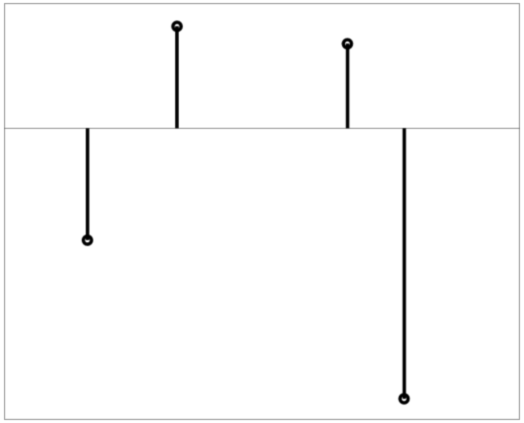

$$\mu \in \mathcal{M}(\mathcal{X})$$

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$$\Phi\mu = \int \varphi(x) d\mu(x)$$

- Fourier

$$\varphi(x) = [e^{2i\pi xk}]_{|k|\leq f_c} \in \mathbb{C}^{2f_c+1}$$

- Convolution

$$\varphi(x) = f(x - \cdot) \in L^2(\mathcal{X})$$

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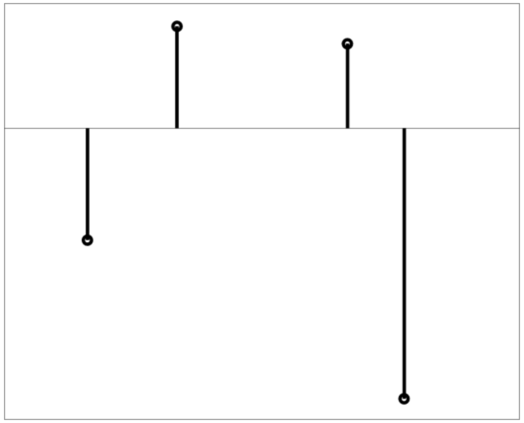

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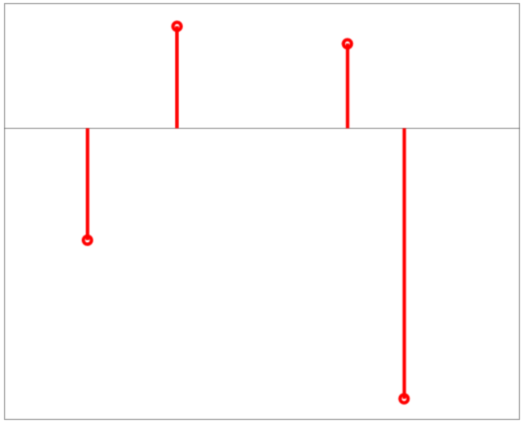

$\tilde{\mu}$

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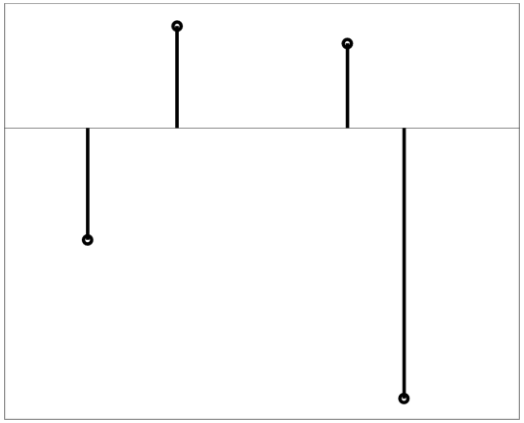

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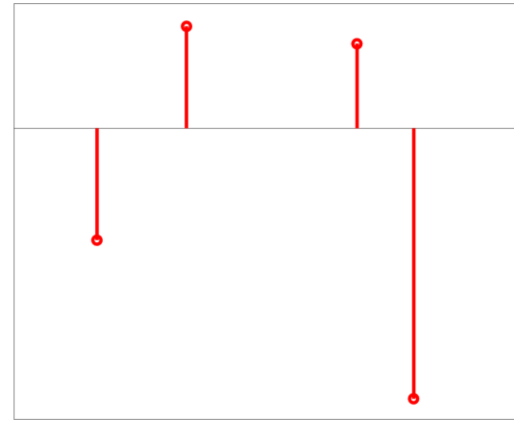

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- **Signal**: Radon measure

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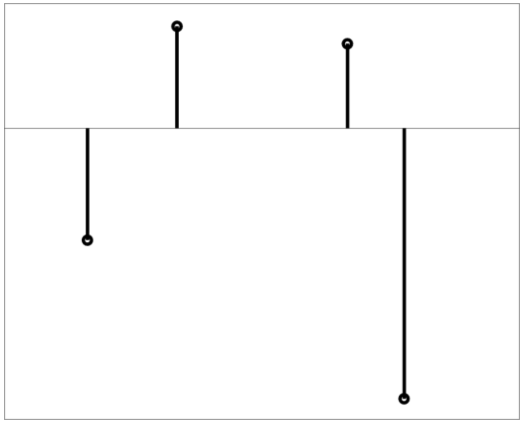

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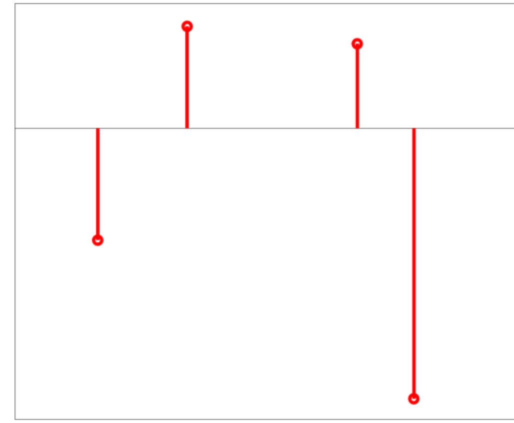

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- **Signal**: Radon measure
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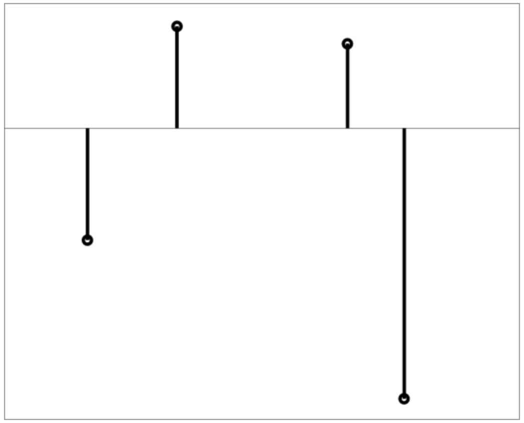

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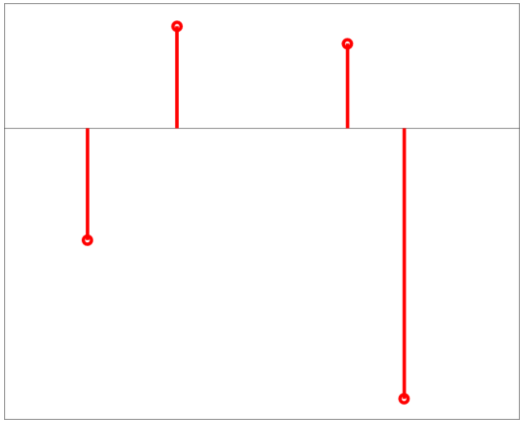

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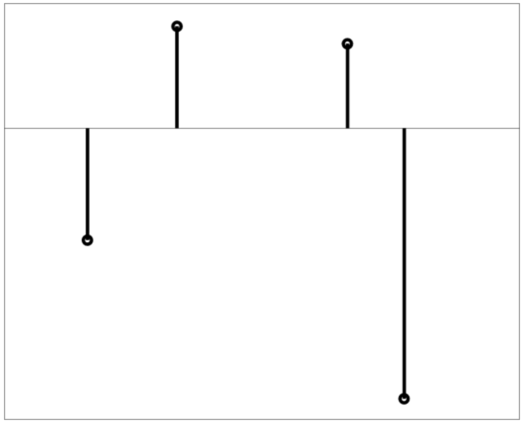

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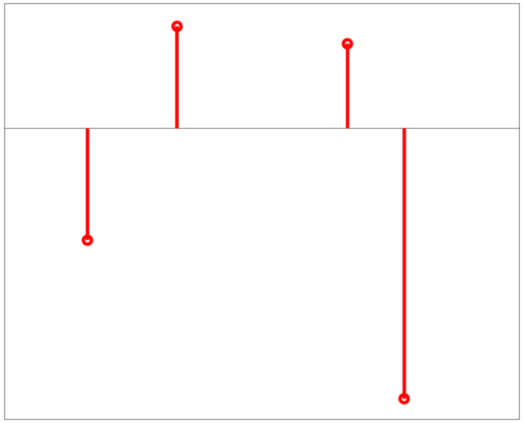

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See Keriven 2017, Gribonval 2017

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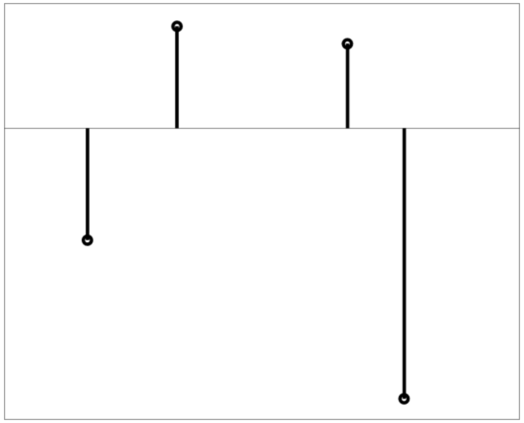

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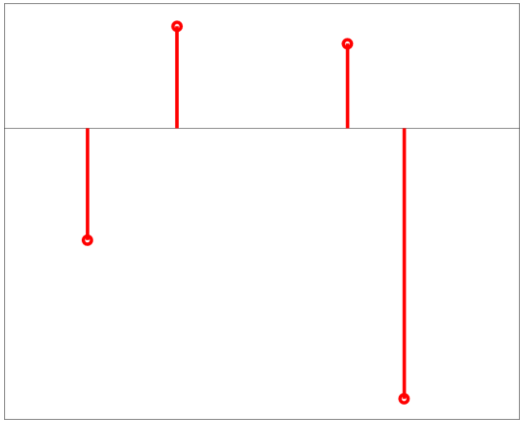

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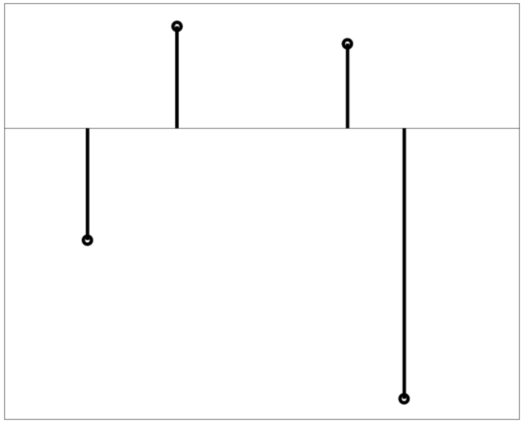

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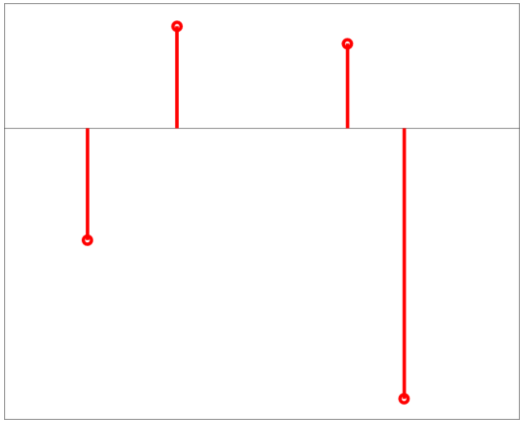

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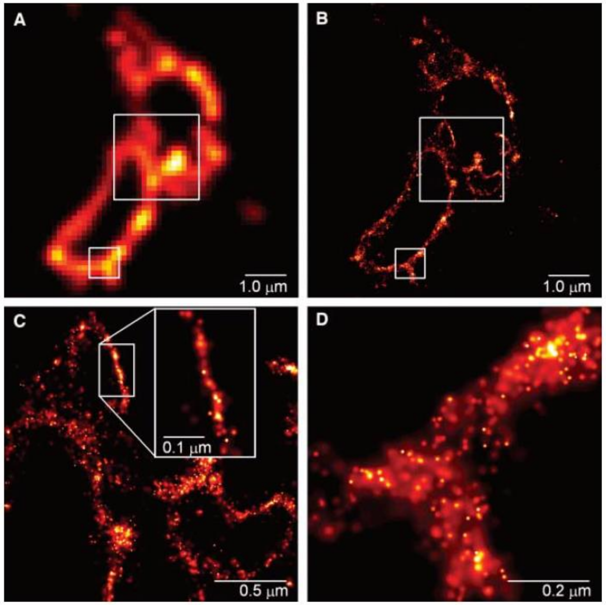
Other approaches: « Prony-like » ESPRIT, MUSIC… (but only 1d noiseless Fourier)

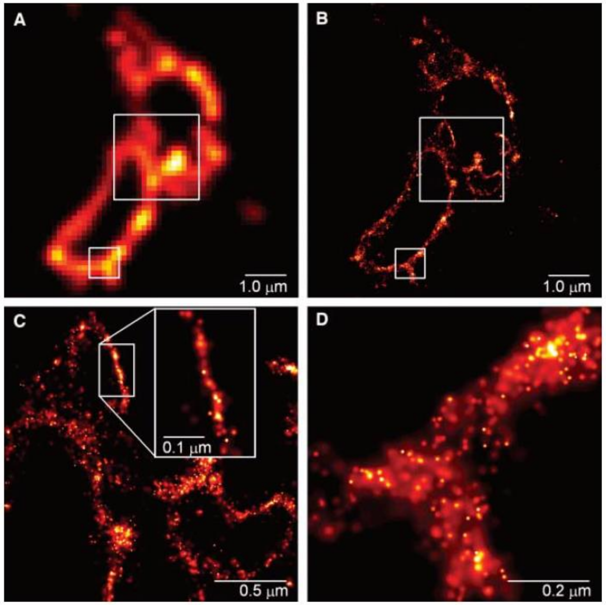
Example of applications



Fluorescence microscopy

[Betzig 2006]

Example of applications

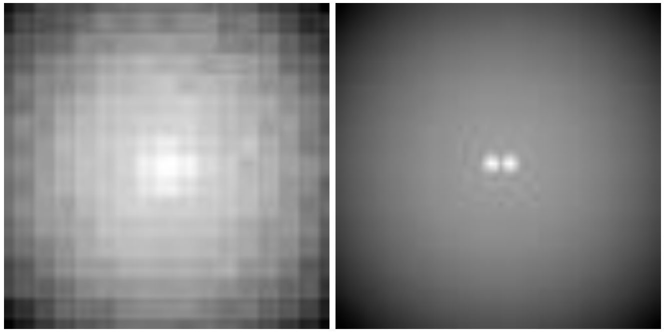


Fluorescence microscopy

[Betzig 2006]

Astronomy

[Puschmann 2017]



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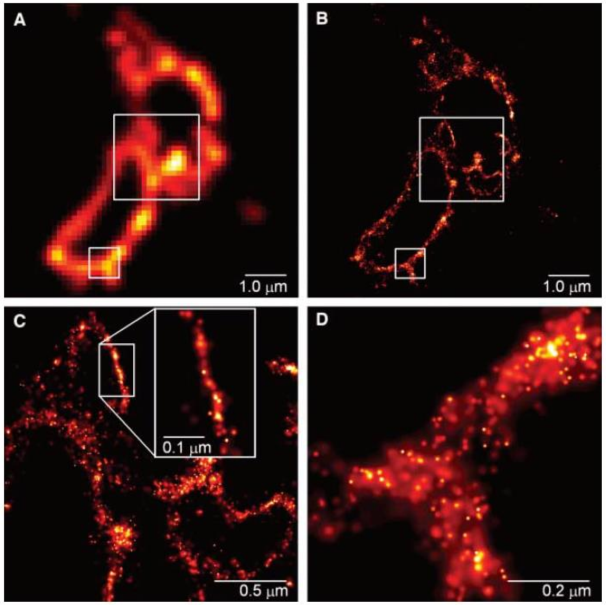

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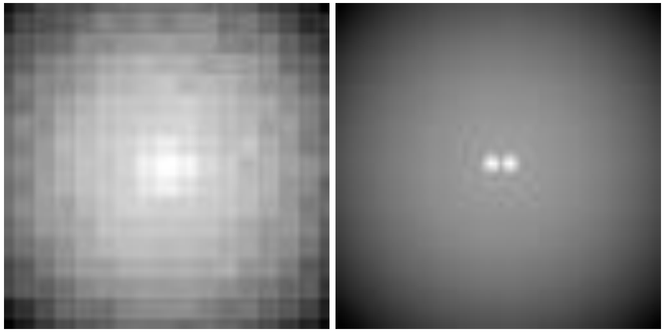
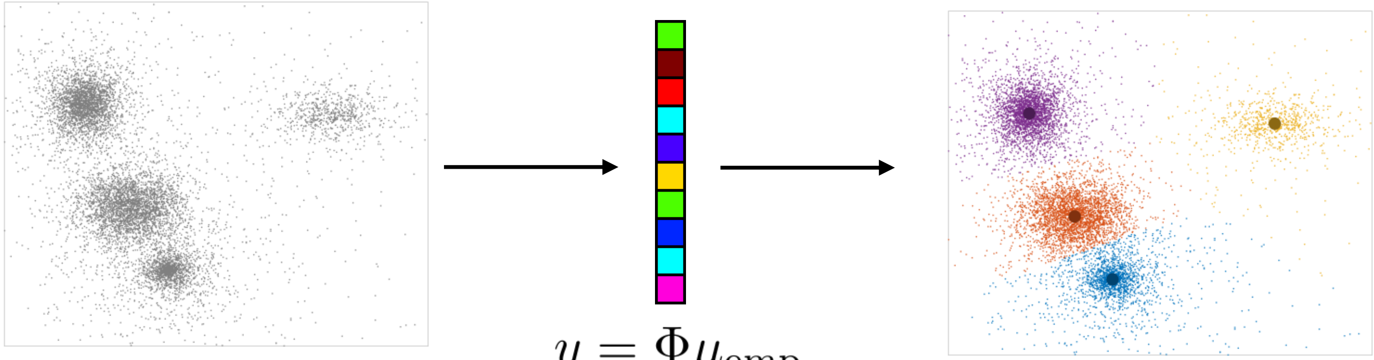


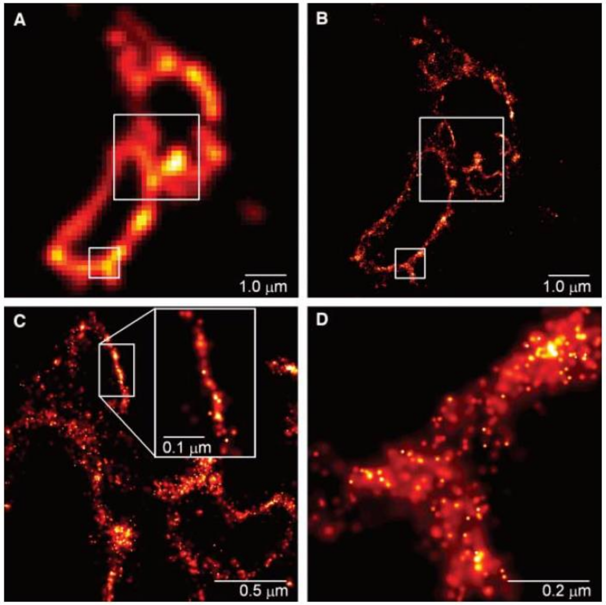
Compressive k-means (GMM…)

[Keriven 2017]



$$y = \Phi \mu_{\mathrm{emp.}}$$

Example of applications



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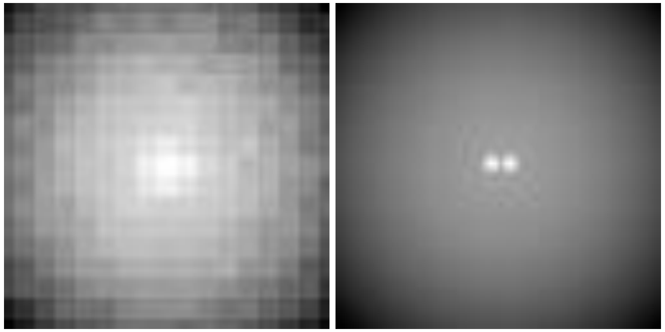

- Neuro-imaging with EEG *[Gramfort 2013]*
- 1-layer neural network *[Bach 2017]*
- Radar
- Geophysics
- …

Compressive k-means (GMM…)

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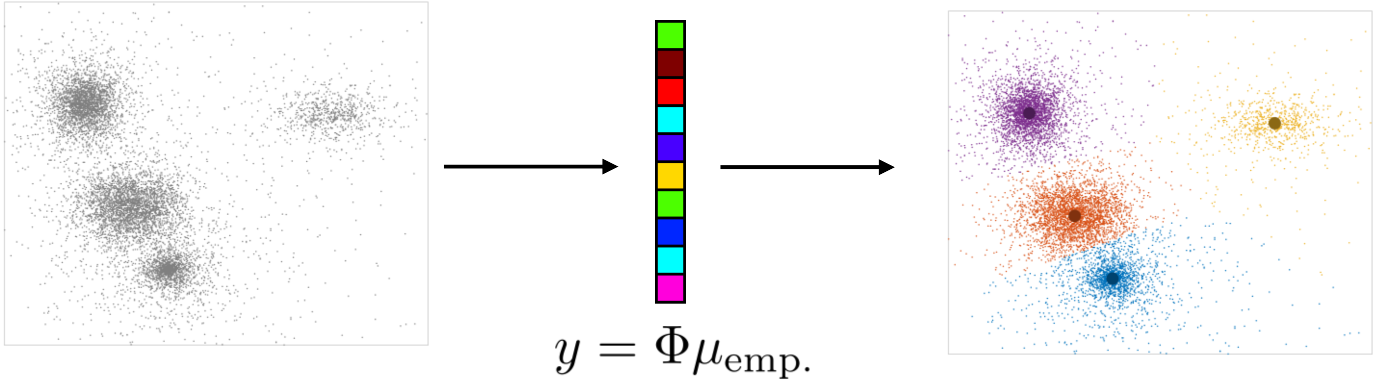
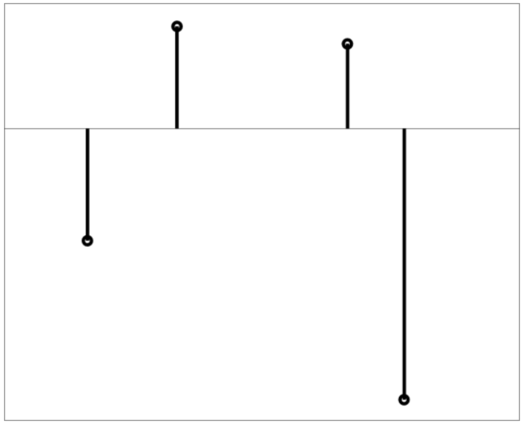


A seminal result *[Candès, Fernandez-Granda 2012]*



$$\mu \in \mathcal{M}(\mathbb{T})$$

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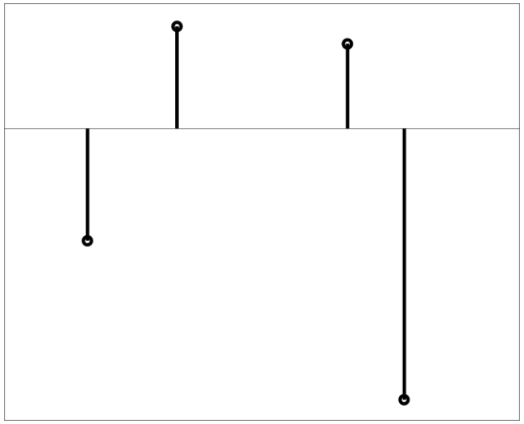

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→

First Fourier coefficients

$$[\int e^{2i\pi x k} d\mu(x)]_{|k| \leq f_c}$$

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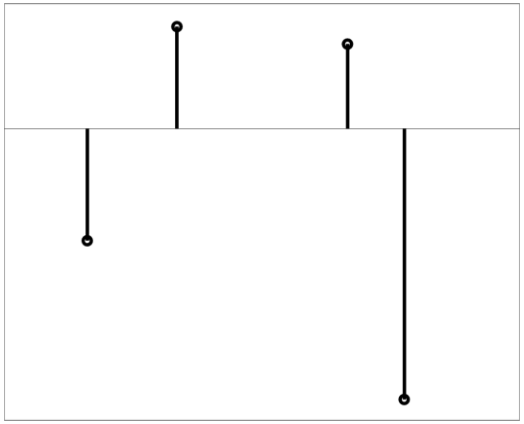

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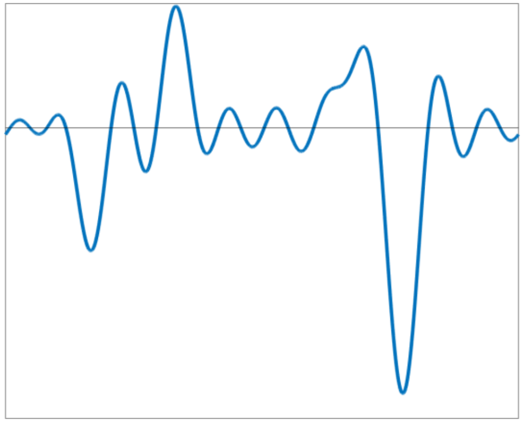
Inverse Fourier



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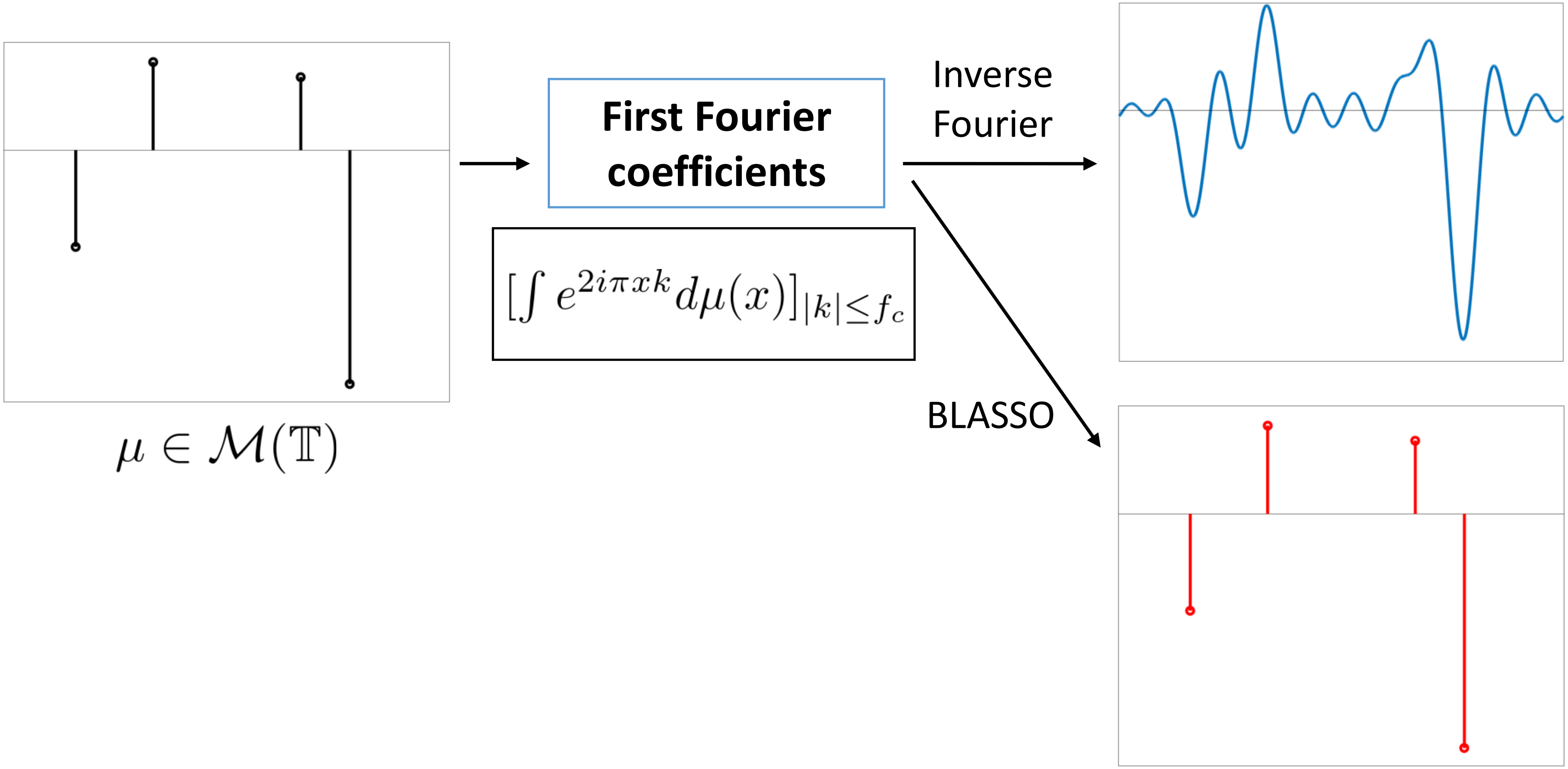

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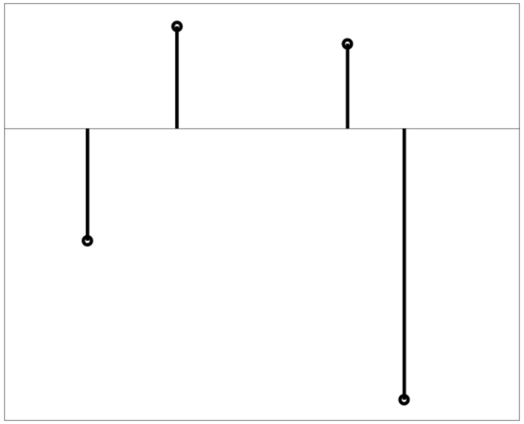

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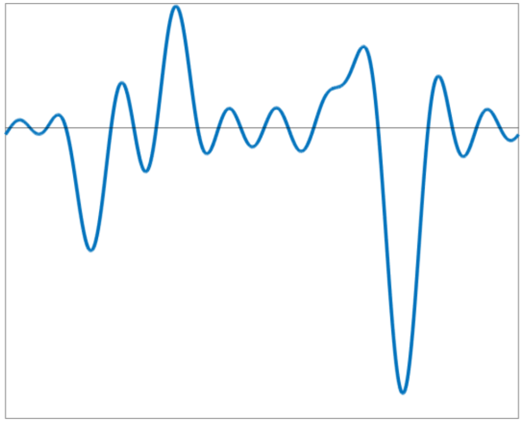

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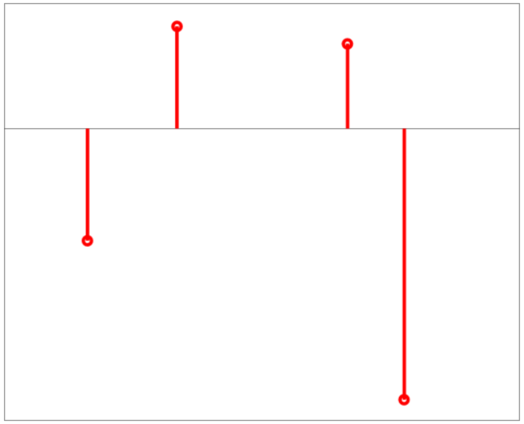

- Minimal separation $\Delta \geq \mathcal{O}(1/f_c)$

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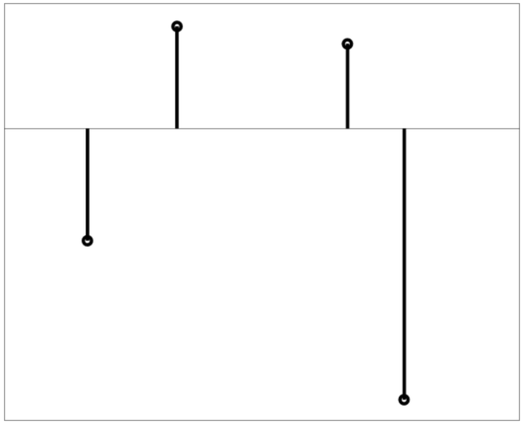

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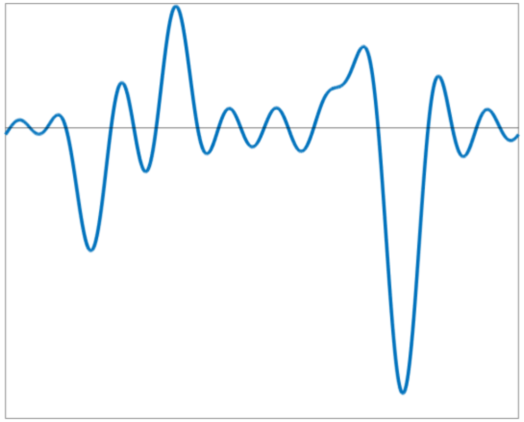

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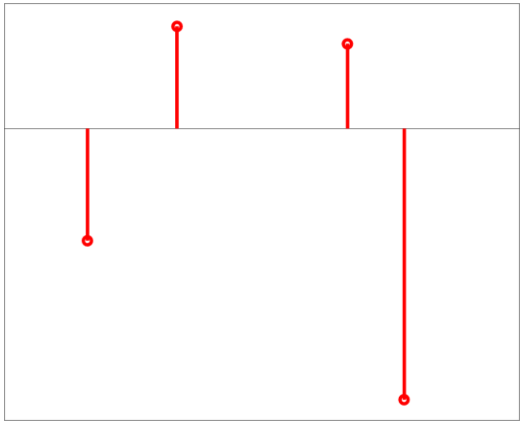

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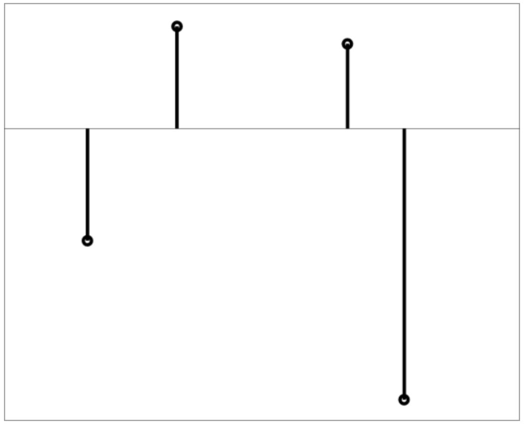

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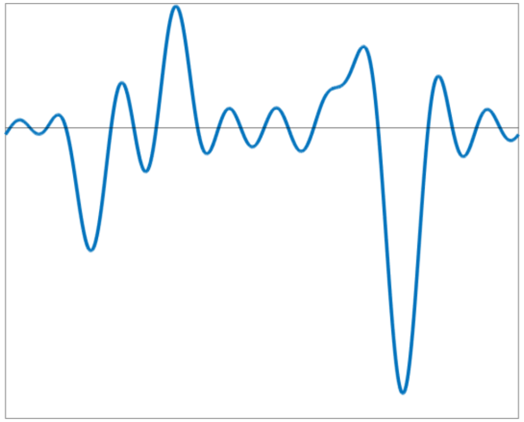

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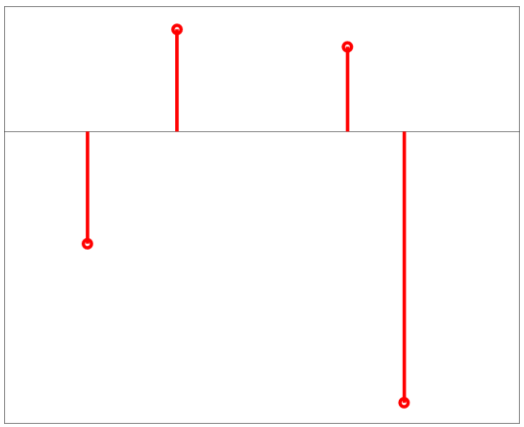

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- **Regular Fourier** on the Torus
- **Reconstruction**: formulated as **SDP** (other: Frank-Wolfe, greedy, Prony-like…)

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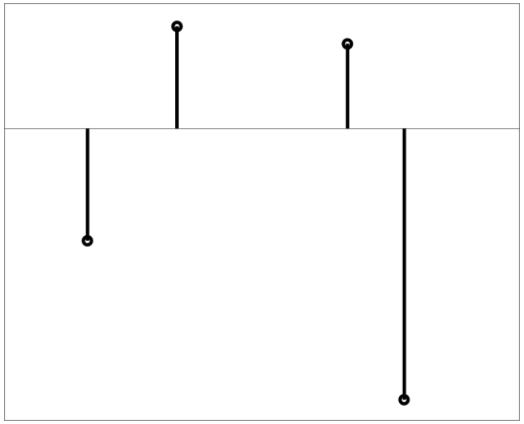

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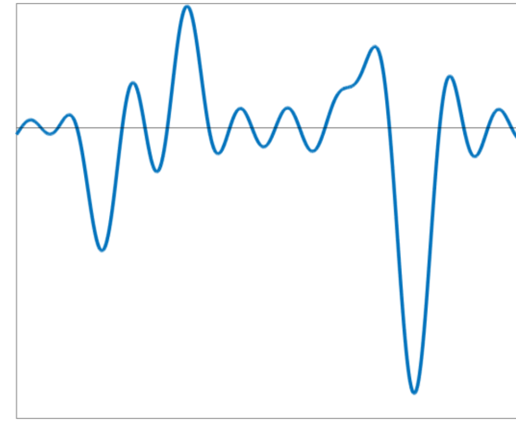
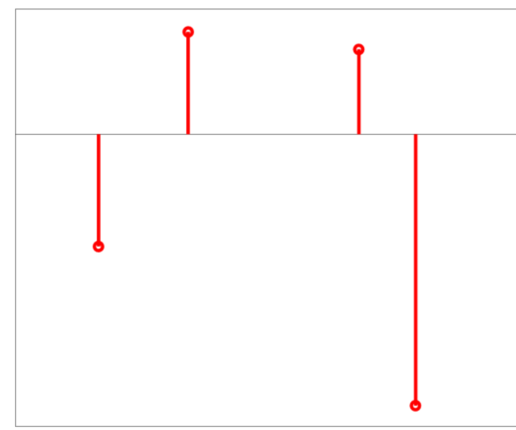
Inverse Fourier →

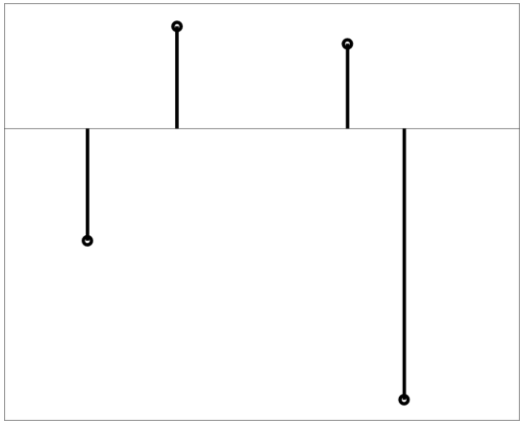


BLASSO ↘



- Minimal separation $\Delta \geq \mathcal{O}(1/f_c)$
- **Regular Fourier** on the Torus
- **Reconstruction**: formulated as **SDP** (other: Frank-Wolfe, greedy, Prony-like…)
- **Noise**: weak convergence (mass of $\tilde{\mu}$ ***concentrated*** around true Diracs)

A seminal result *[Candès, Fernandez-Granda 2012]*



$\mu \in \mathcal{M}(\mathbb{T})$

First Fourier coefficients

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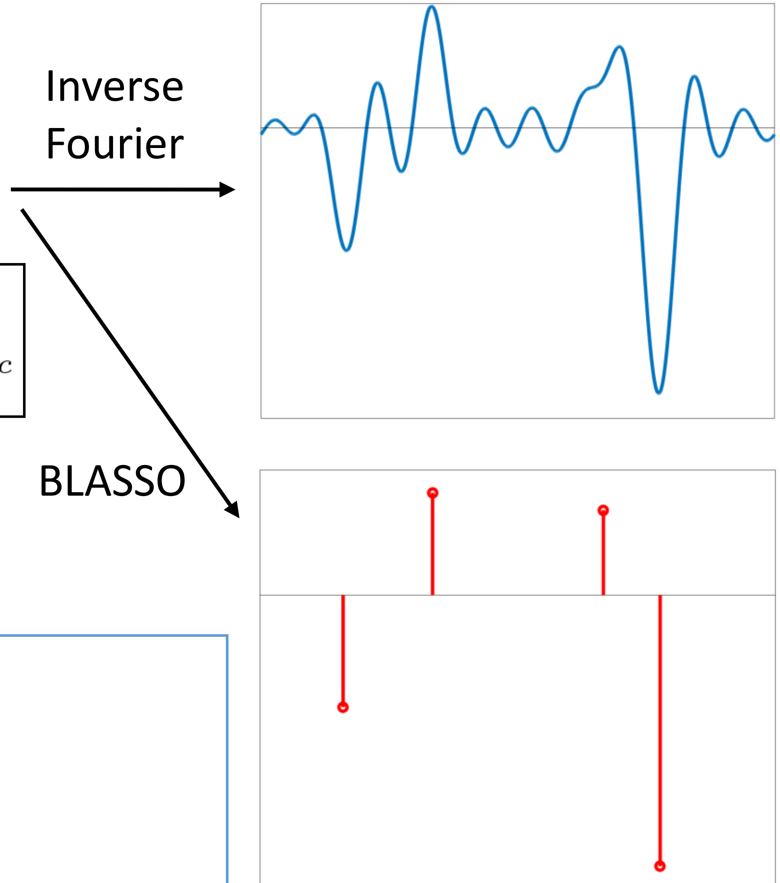

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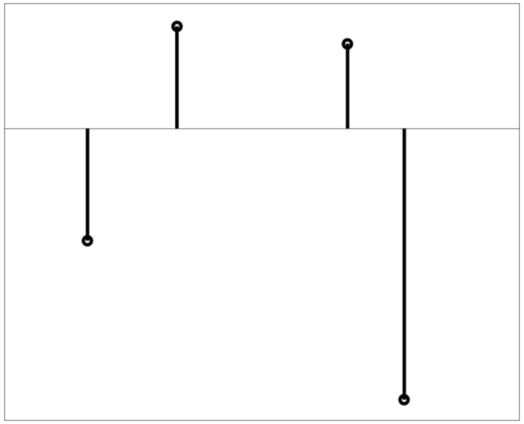
Randomness ? Number of measurements ?

Compressed sensing off-the-grid *[Tang, Recht 2013]*



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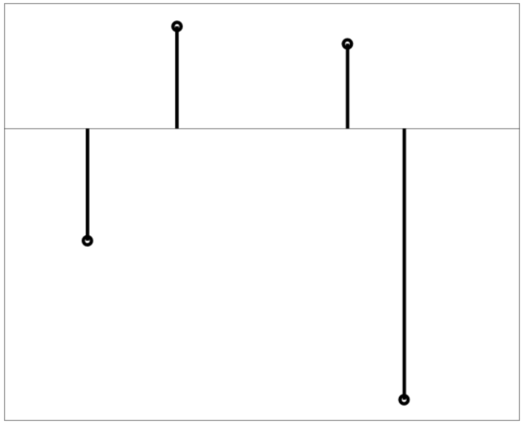

Random Fourier coefficients

$$\left[\int e^{2i\pi x k} d\mu(x)\right]_{k \in M}$$

$M \subset [-f_c, f_c]$ random

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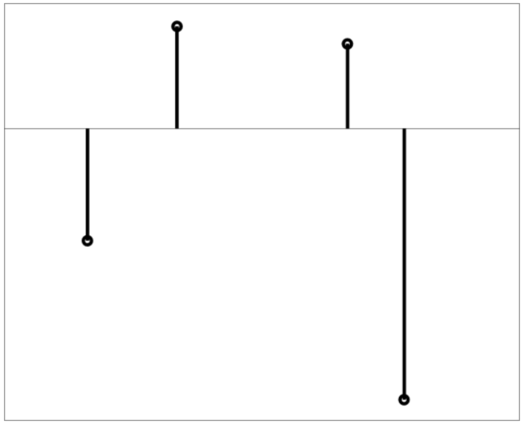

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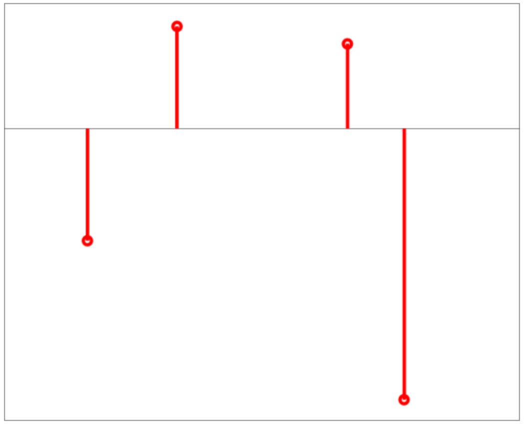

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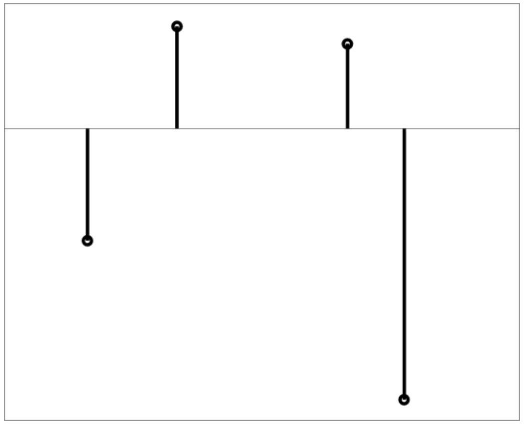

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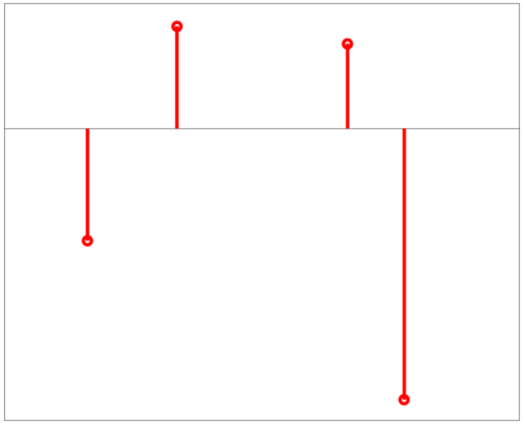

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- **Random signs assumption**

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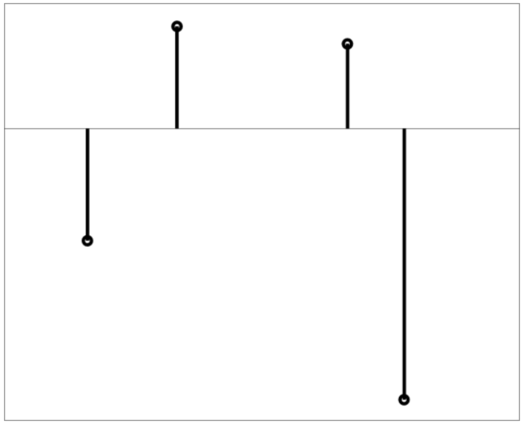

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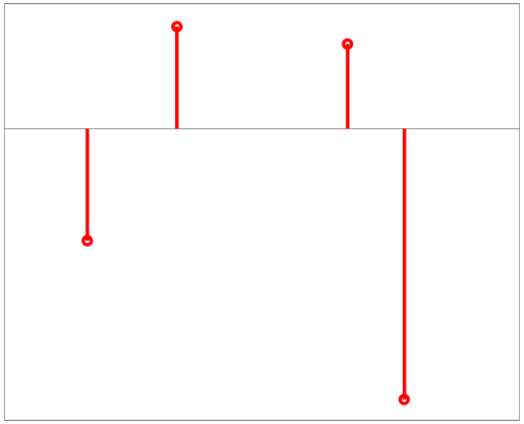

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Questions:

- More general sampling scheme ?
- Multi-dimensional result ?
- **Get rid of random signs** ?

Outline

① Background on dual certificates

② Compressive off-the-grid recovery

③ Conclusion, outlooks

Dual certificates

Measurements

$$\Phi\mu = \int \varphi(x) d\mu(x)$$

$\varphi : \mathcal{X} \to \mathcal{H}$ Hilbert space

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Dual certificate **(noiseless case)**

μ_0 solution of $\min_{\Phi\mu = y} |\mu|(\mathcal{X})$ $\iff \mathrm{Im}(\Phi^\star) \cap \partial|\mu_0|(\mathcal{X}) \neq \emptyset$

What does it look like ?

What is a dual certificate ?

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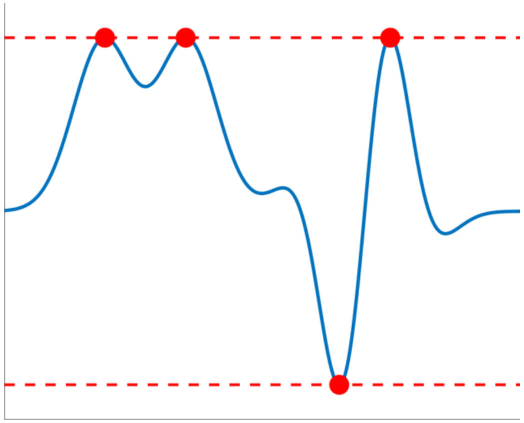

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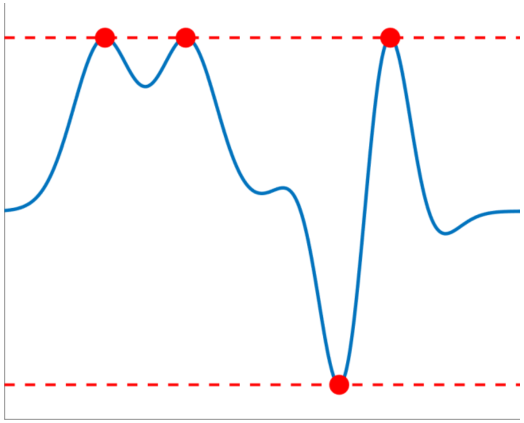
Non-degenerate dual certif.

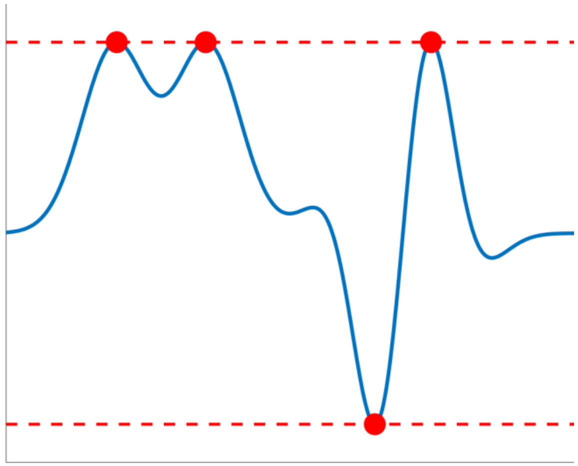
$$|\eta(x)| < 1$$

$$\mathrm{sign}(a_i)\nabla^2\eta(x_i) \prec 0$$



Ensures **uniqueness** and **robustness**…

Recovery results



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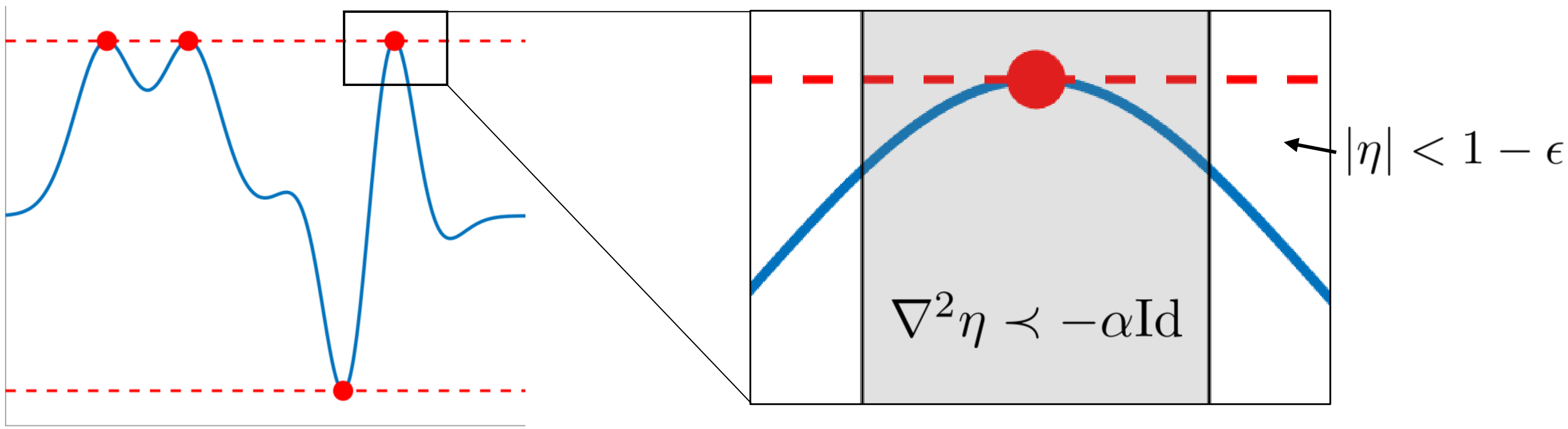

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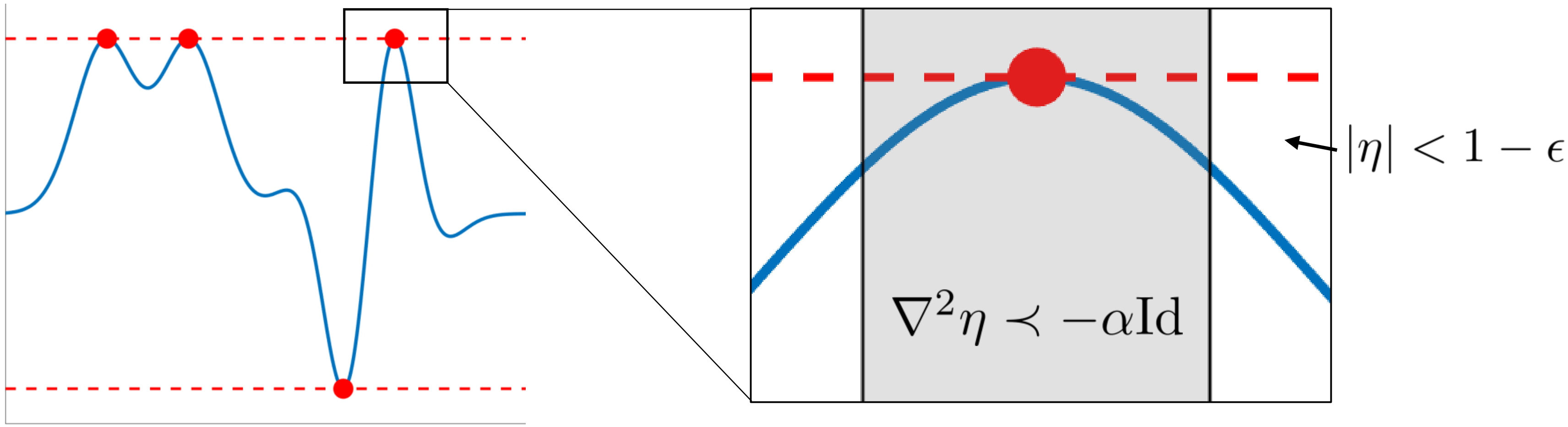

Theorem *(refinement of [Azaïs et al. 2015])*

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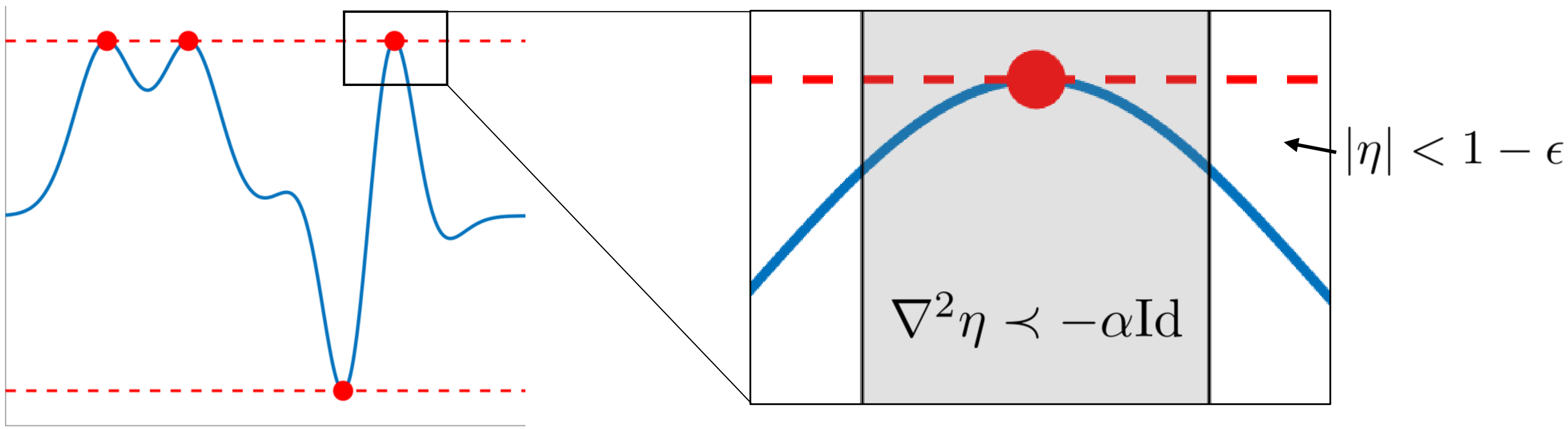

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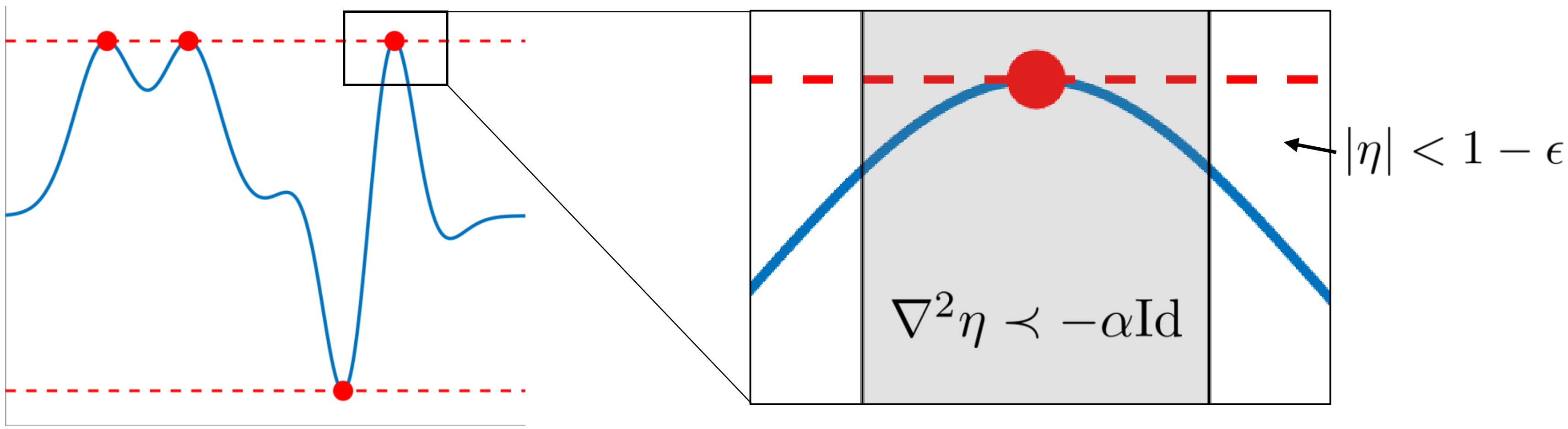

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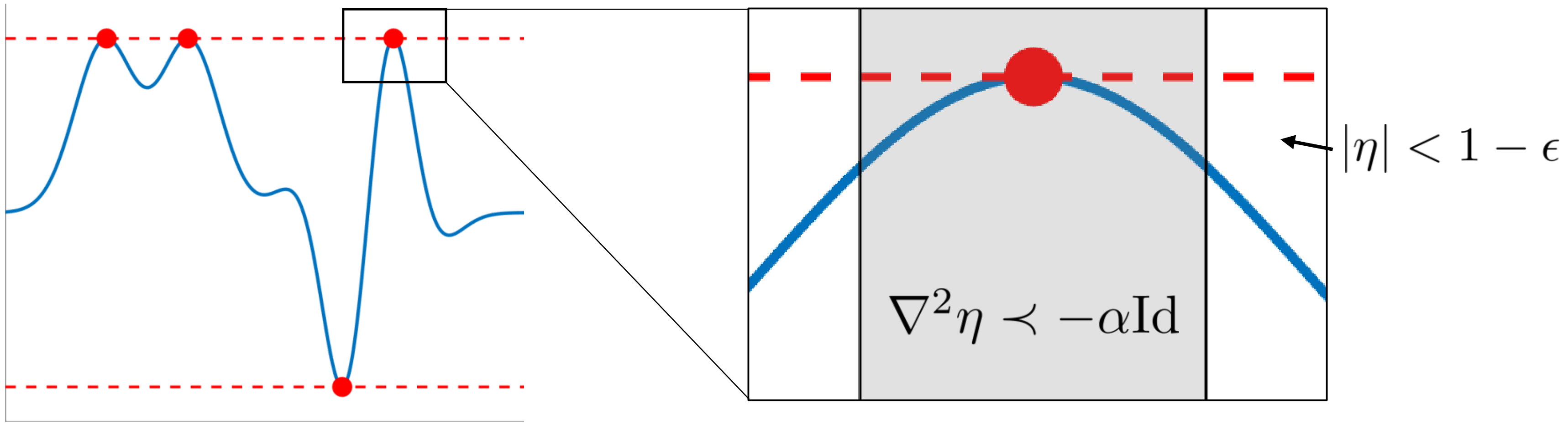

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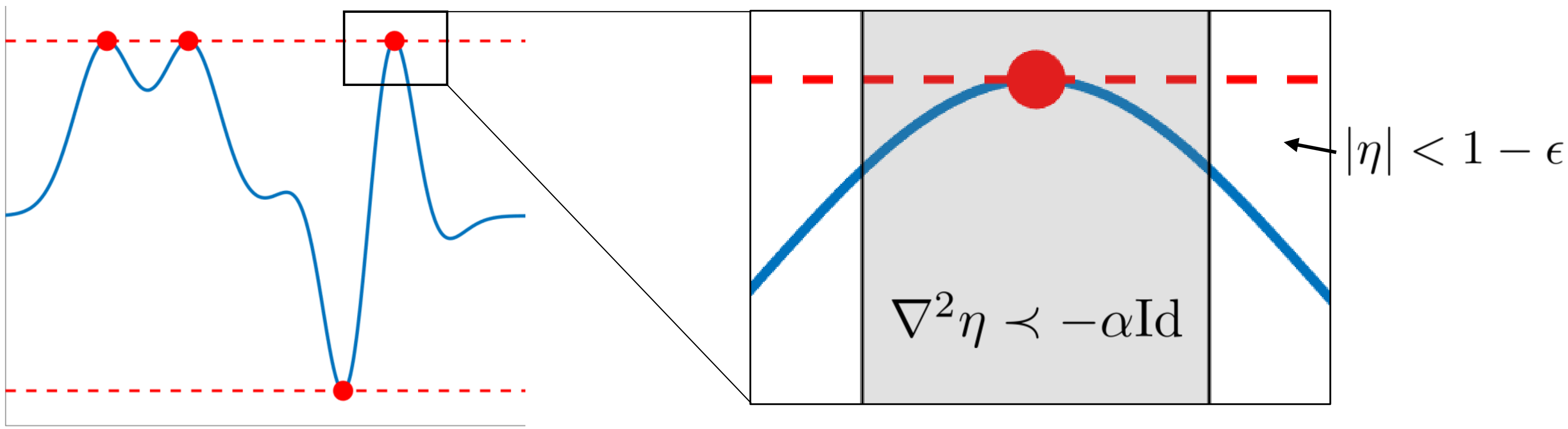

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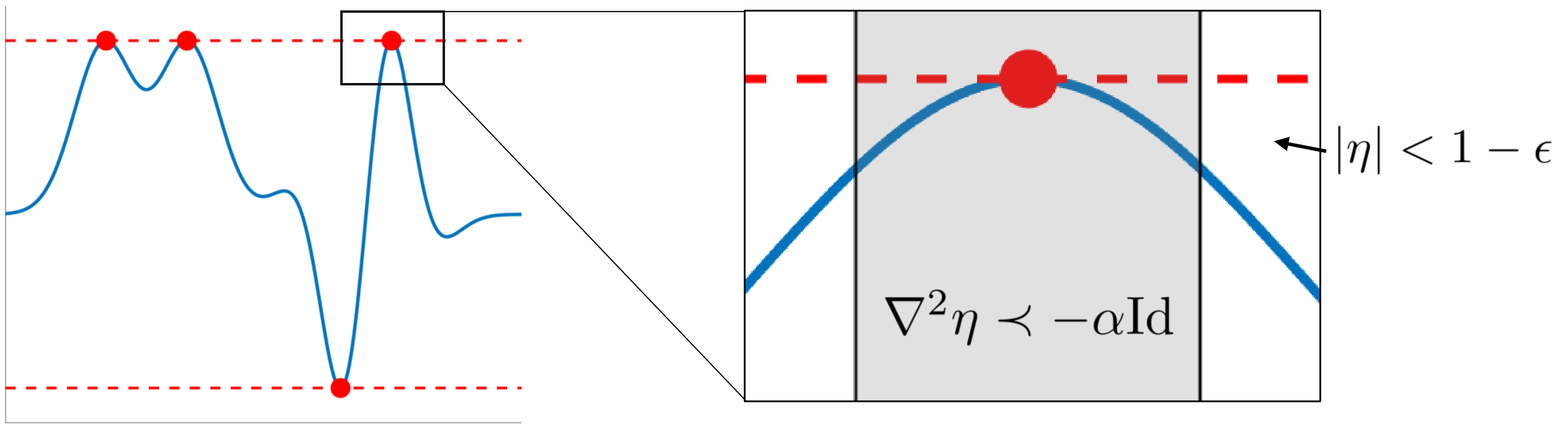

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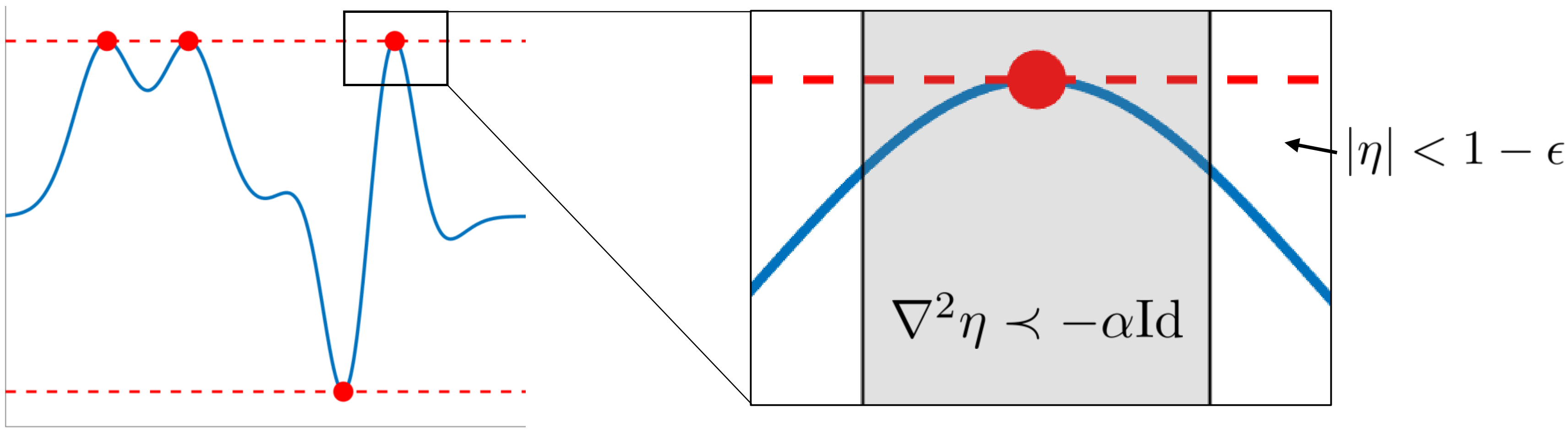

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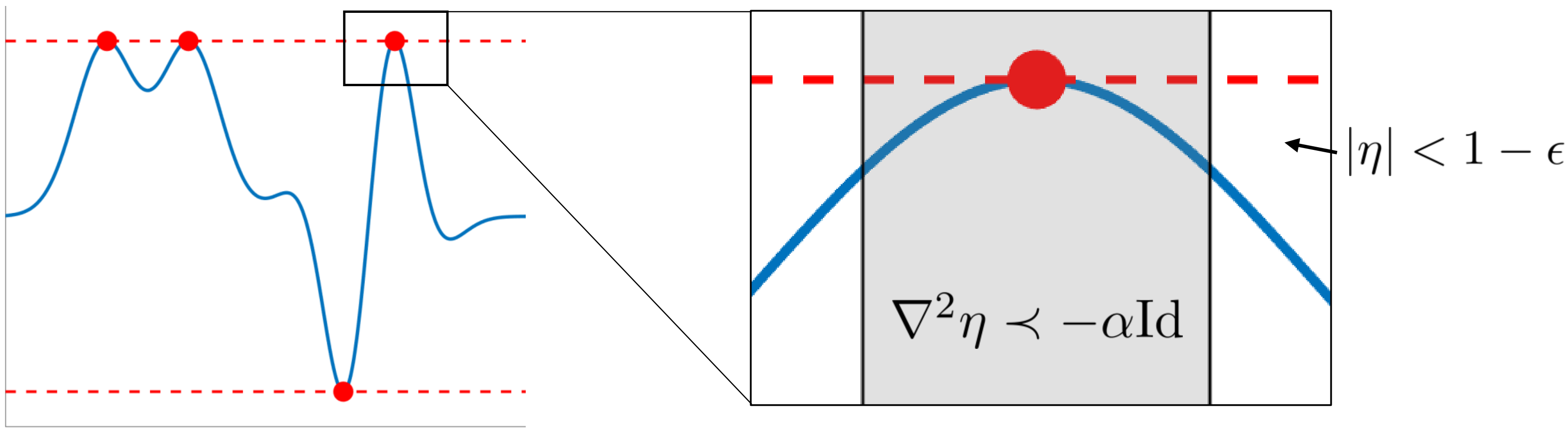

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- $\tilde{\mu}$ **is sparse, with the right number of components**
- $(\tilde{a}_i, \tilde{x}_i) \xrightarrow[\|e\| \to 0]{} (a_i, x_i)$

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Low-dim, random

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Ex: full Fourier (high-dim), full convolution (infinite-dim)

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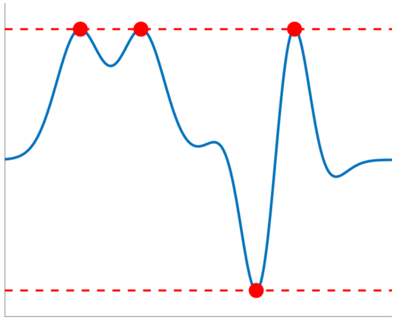

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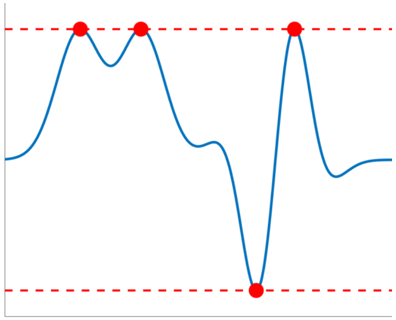

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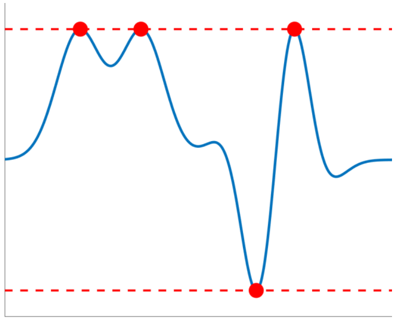

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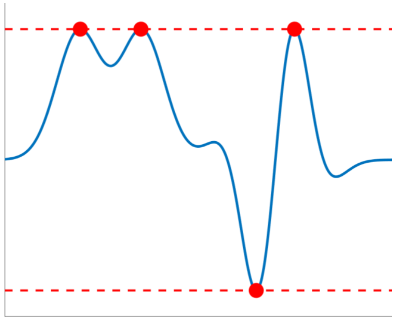

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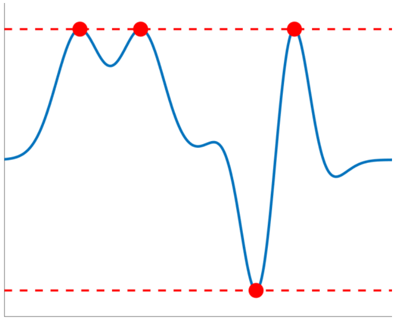

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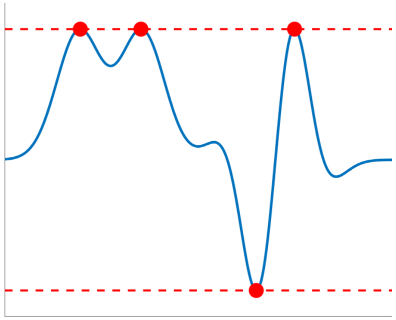

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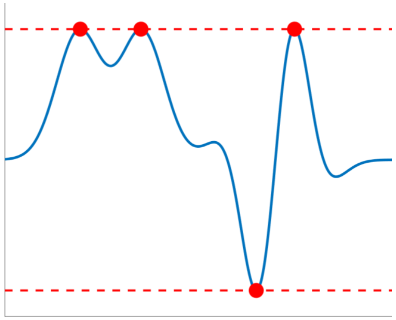

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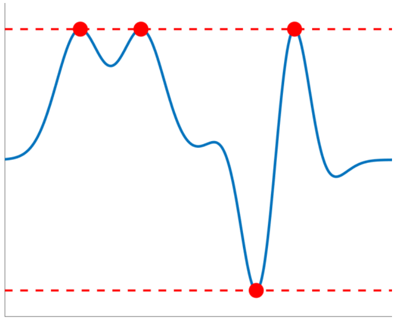
$$\eta_{\text{full}} \in \text{Span}\left\{\kappa_{\text{full}}(x_i, \cdot), \partial_1 \kappa_{\text{full}}(x_i, \cdot)\right\} \subset \text{Im}(\Phi^\star_{\text{full}})$$



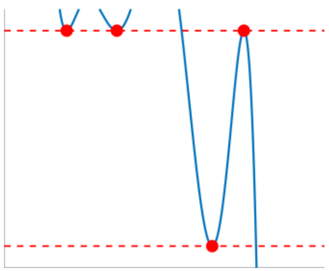
Step 2: Use ***Random Features*** on full kernel to define sampling

Assuming $\kappa_{\text{full}}(x, x') = \mathbb{E}_{\omega \sim \Lambda} \phi_\omega(x) \overline{\phi_\omega(x')}$

Define $\varphi(x) = \frac{1}{\sqrt{m}} [\phi_{\omega_j}(x)]_{j=1}^m$

Then $\kappa(x, x') = \langle \varphi(x), \varphi(x') \rangle_{\mathbb{C}^m}$

$\approx \kappa_{\text{full}}(x, x')$



m=10

Goal

Goal: random sampling

$$\Phi\mu = \int \varphi(x) d\mu(x)$$

Low-dim, random

Strategy: Start with « high »-dimensional problem

$$\Phi_{\text{full}}\mu = \int \varphi_{\text{full}}(x) d\mu(x)$$

Ex: full Fourier (high-dim), full convolution (infinite-dim)

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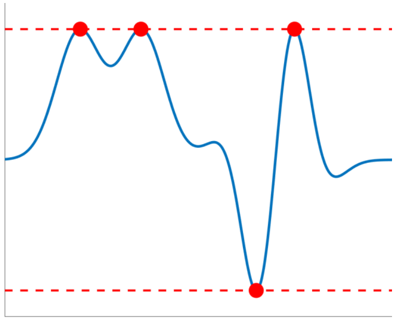

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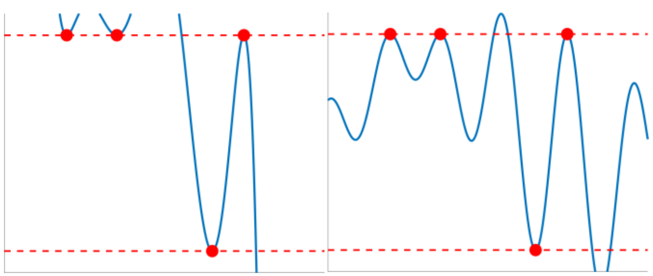

m=10 *m=50*

Goal

Goal: random sampling

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Low-dim, random

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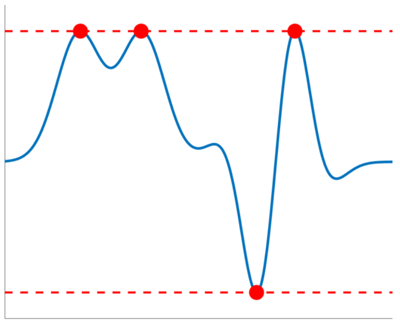

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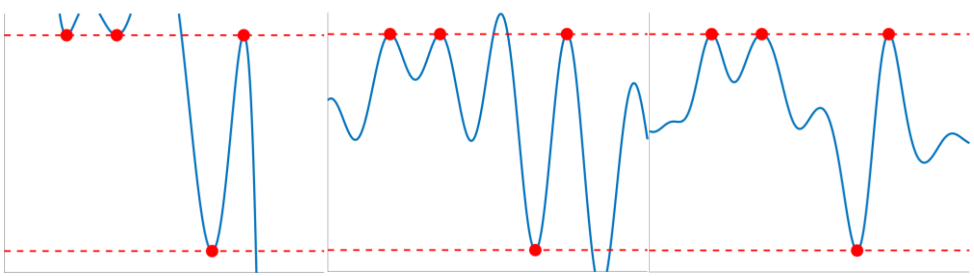

m=10 *m=50* *m=500*

Step 1: Acceptable full kernels

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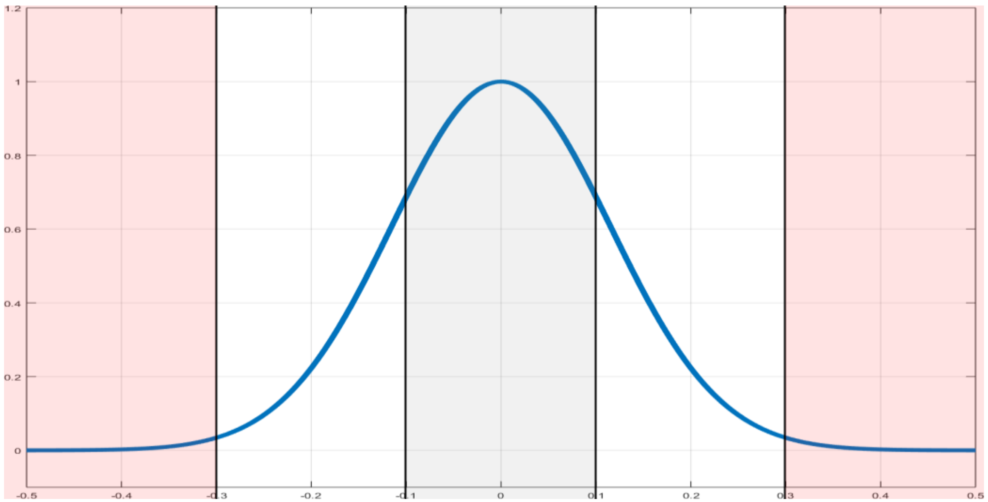

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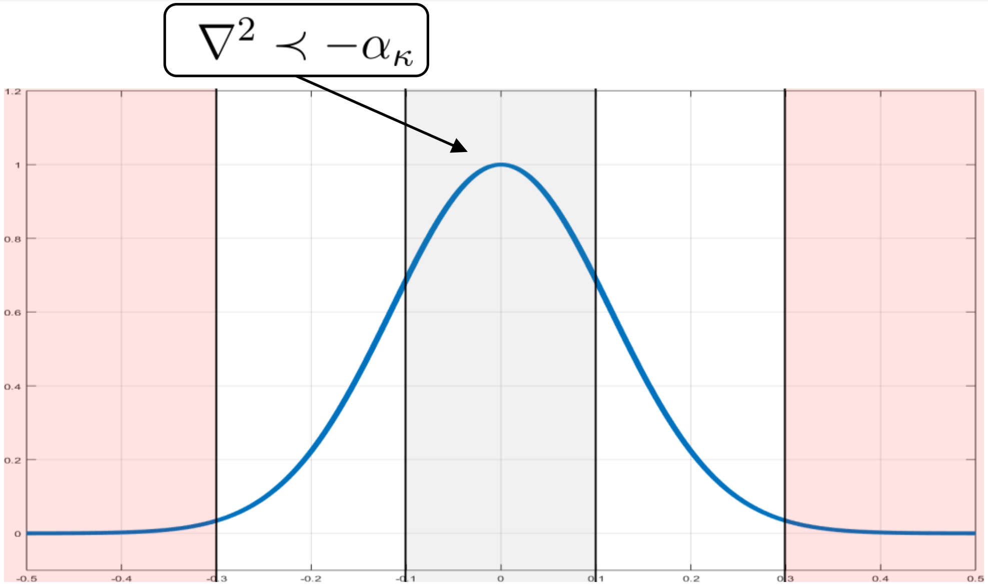

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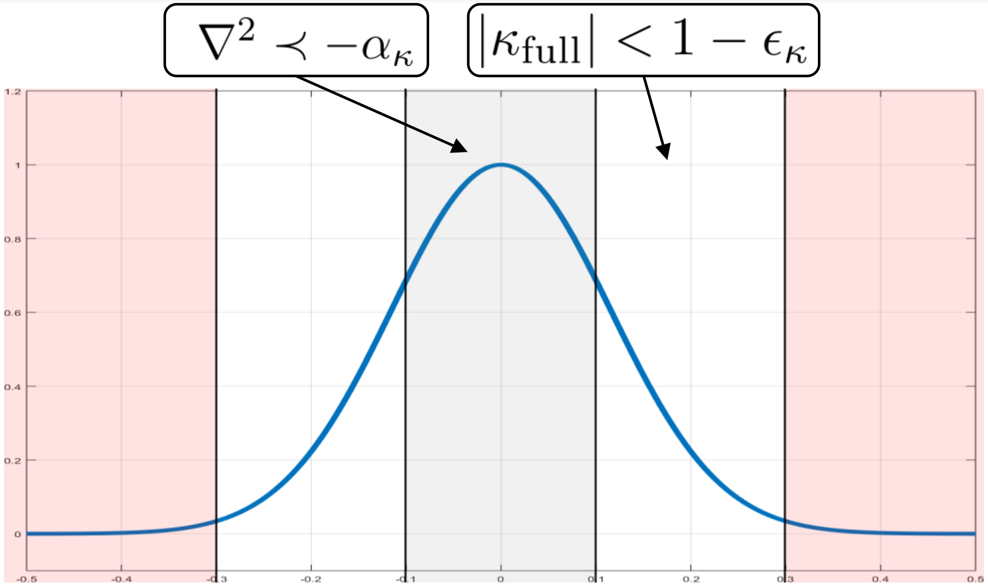

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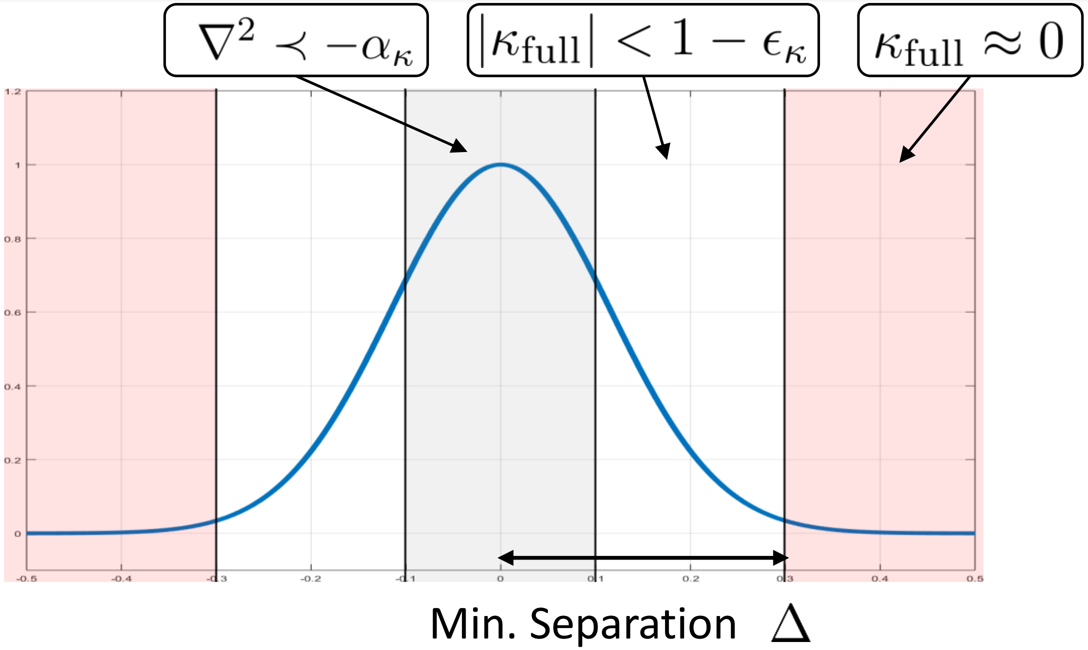

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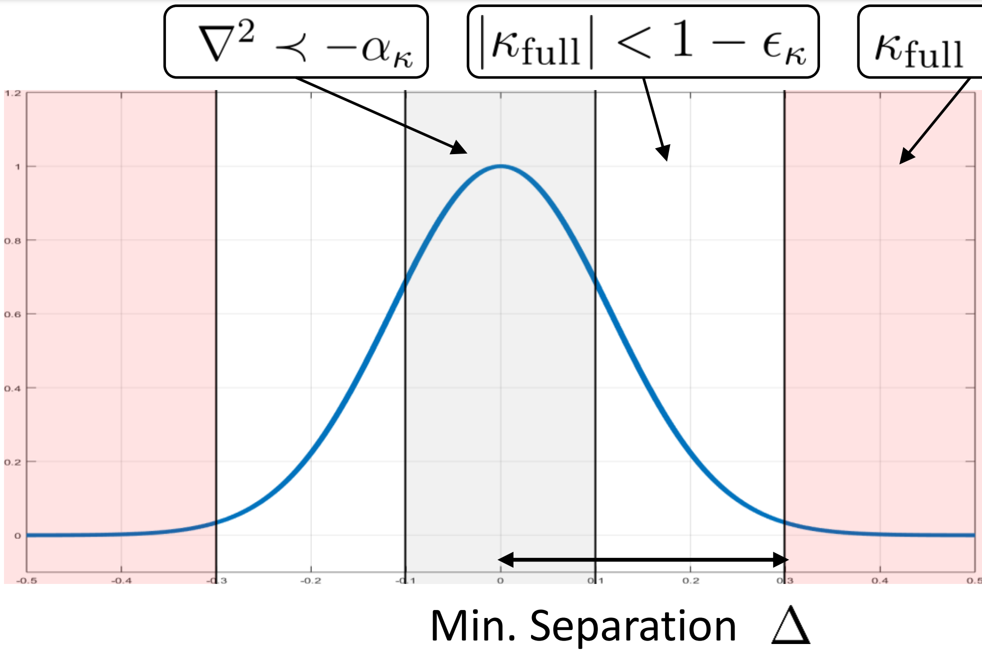
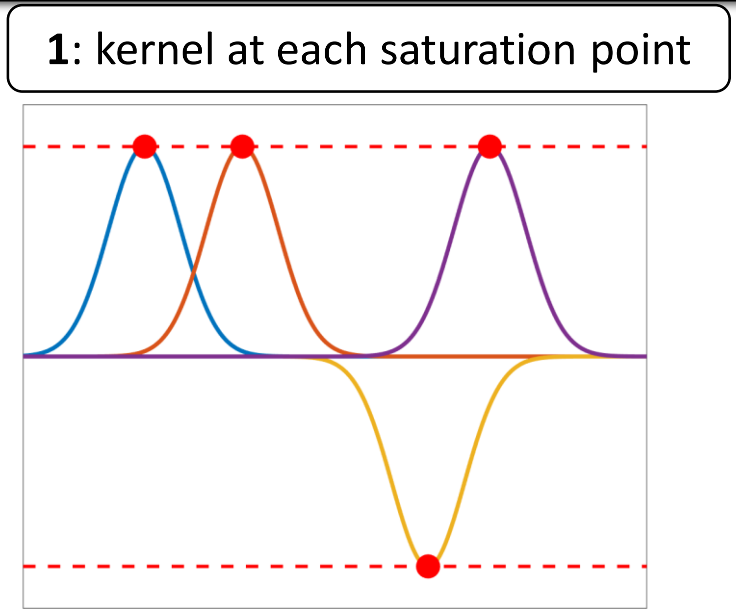


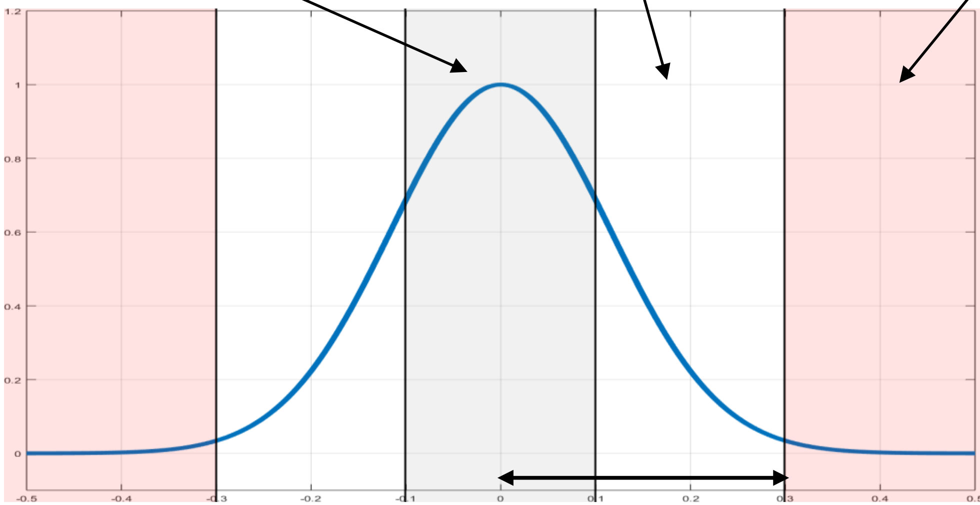


Step 1: Acceptable full kernels

$\nabla^2 \prec -\alpha_\kappa$

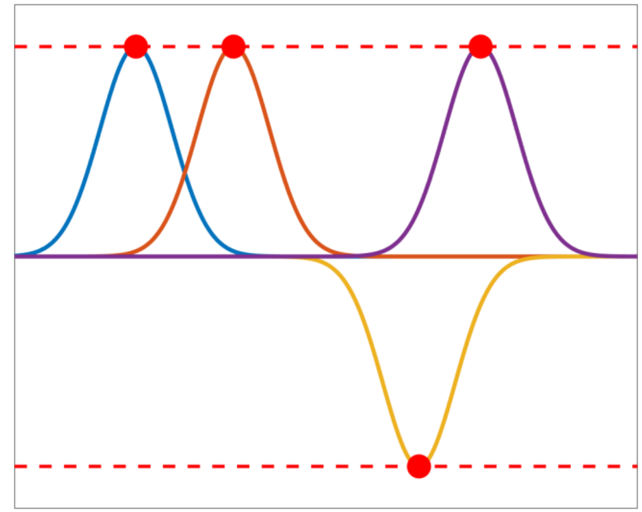
$|\kappa_{\text{full}}| < 1 - \epsilon_\kappa$

$\kappa_{\text{full}} \approx 0$

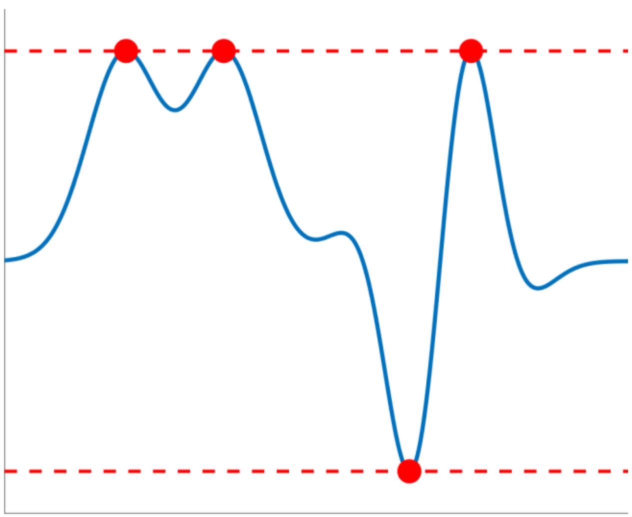


Min. Separation Δ

1: kernel at each saturation point



2: ***Small*** adjustements: minimal separation



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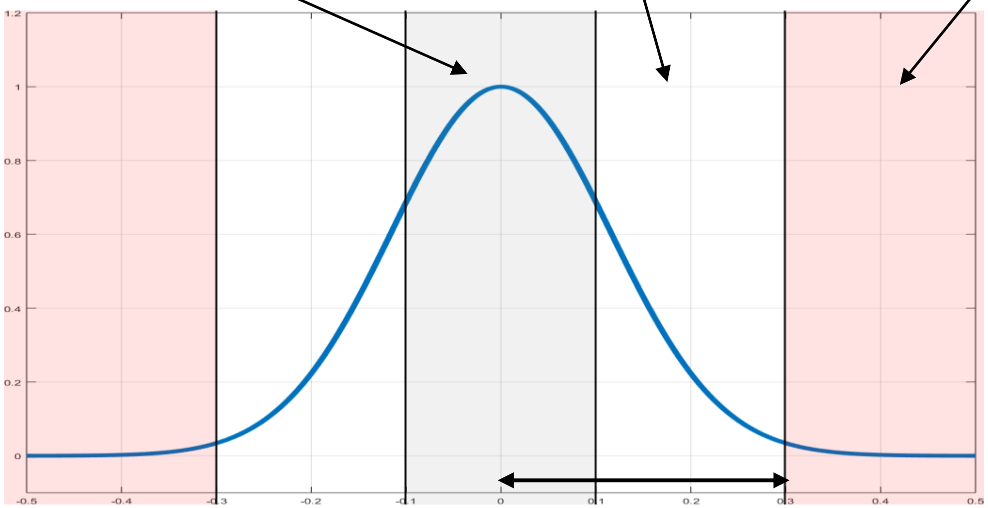

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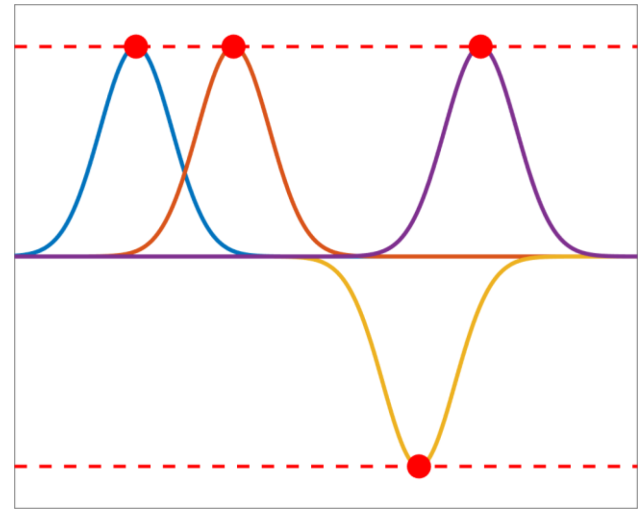

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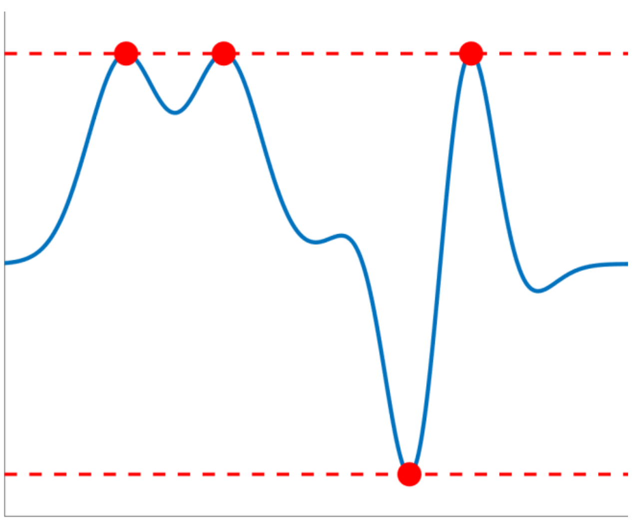

- ***Multi-d square Féjer*** kernel (regular Fourier on Torus)

$$\Delta = \mathcal{O}(s^{\frac{1}{4}} d^{\frac{1}{2}} f_c^{-1})$$

- ***Multi-d Gaussian*** kernel

$$\Delta = \mathcal{O}(\sigma^{-1}\sqrt{\log(s) + \log(d)})$$

Step 2: sampling

Thm: Ideal scaling in sparsity: infinite-dim. golfing scheme

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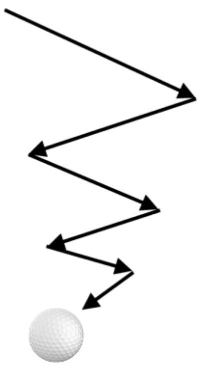

$$m = m_1 + m_2 + m_3 + \dots$$

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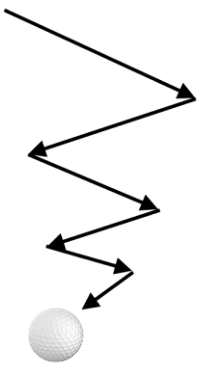

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With random signs

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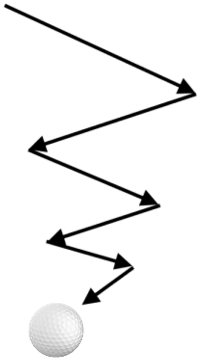

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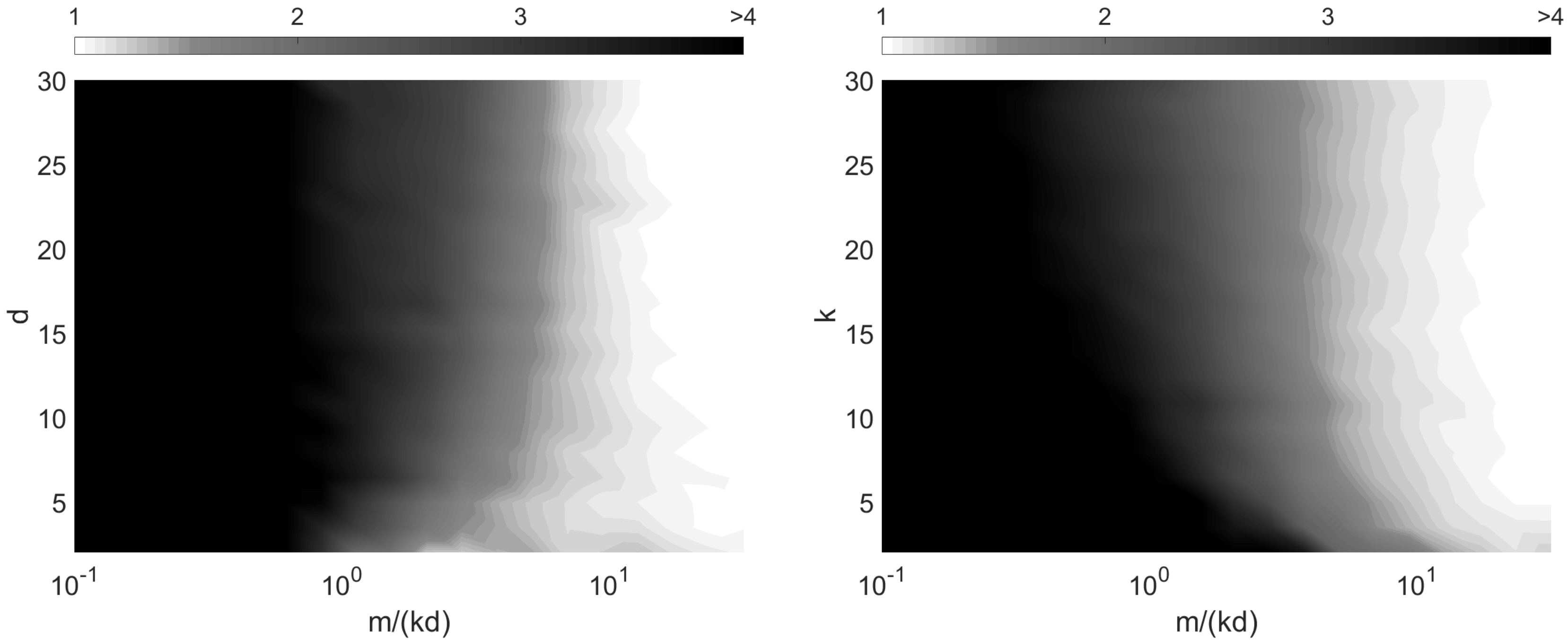
Fun application: convex approach for automatic estimation of number of components in a GMM

Number of measurements in practice ?

Compressive k-means *[Keriven 2017]*



Relative number of measurements m/(sd)

Outline

① Background on dual certificates

② Compressive off-the-grid recovery

③ Conclusion, outlooks

Summary, outlooks

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  - Beyond Fourier on the Torus (« acceptable » kernels)
  - Multi-d
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  - Support recovery when random signs (or quadratic number of measurements)

Summary, outlooks

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  - Multi-d
  - No need for random signs for basic recovery result
  - Support recovery when random signs (or quadratic number of measurements)

- **Outlooks**
  - Other kernels, *very different from translation-invariant*
  - More quantified treatment of dimension
  - Other practical applications (eg 1-layer neural networks with continuum of neurons *[Bach 2017]*)

Thank you !

Poon, Keriven, Peyré. **A Dual Certificates Analysis of Compressive Off-the-Grid Recovery.** *Preprint arxiv:1802.08464*

Code: sketchml.gforge.inria.fr,
github: nkeriven

data-ens.github.io

Enter the data challenges!
Come to the colloquium!
Come to the Laplace seminars!