Fisher metric, support stability and optimal number of measurements in compressive off-the-grid recovery

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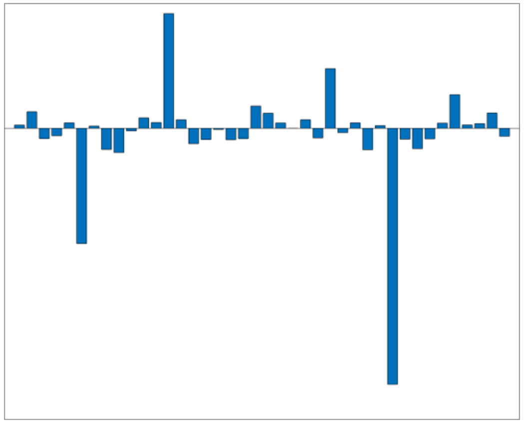
CFM-ENS chair in Data Science

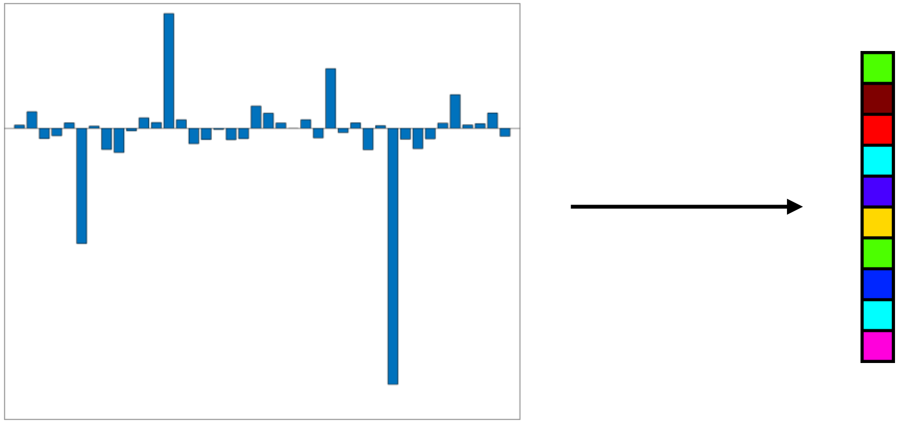
Joint work with **Clarice Poon** (Cambridge Uni.), **Gabriel Peyré** (ENS)

Discrete compressive sensing



$$x \in \mathbb{R}^n$$

Discrete compressive sensing



$$x \in \mathbb{R}^n \qquad\qquad y = Mx + w$$

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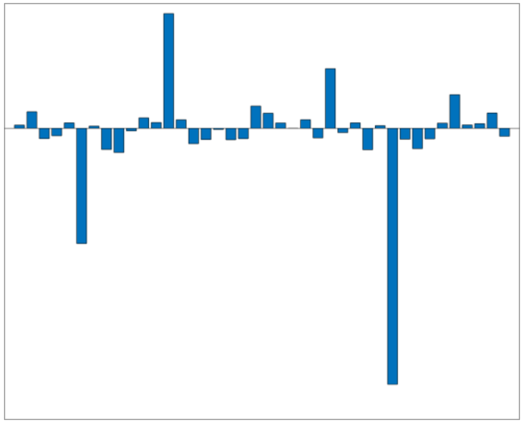

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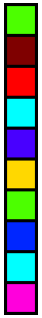

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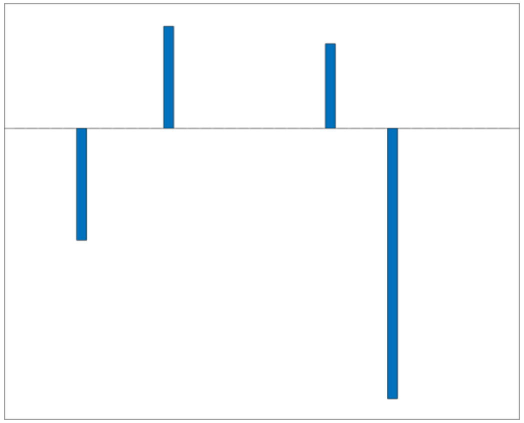
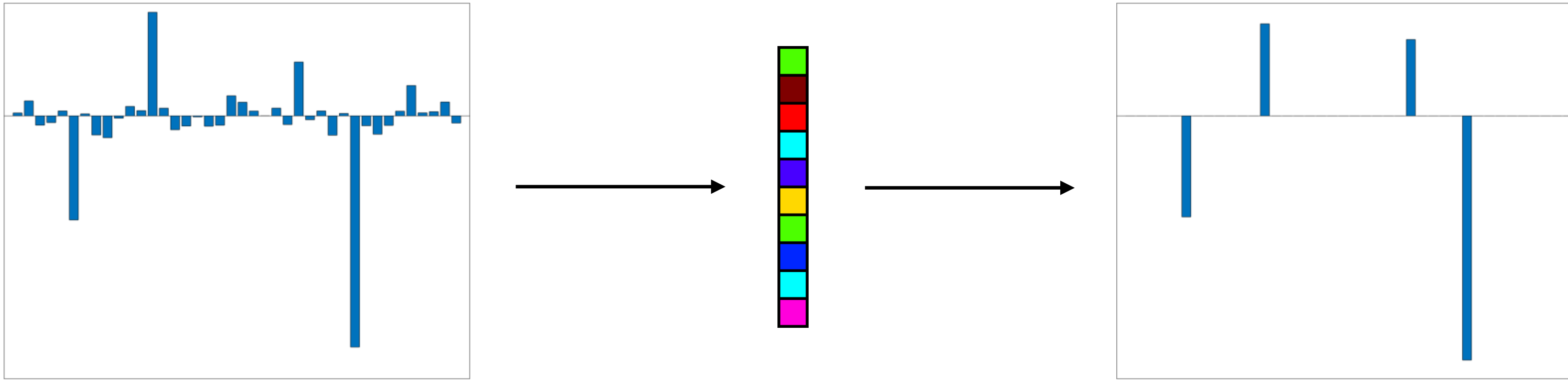


$\tilde{x}$

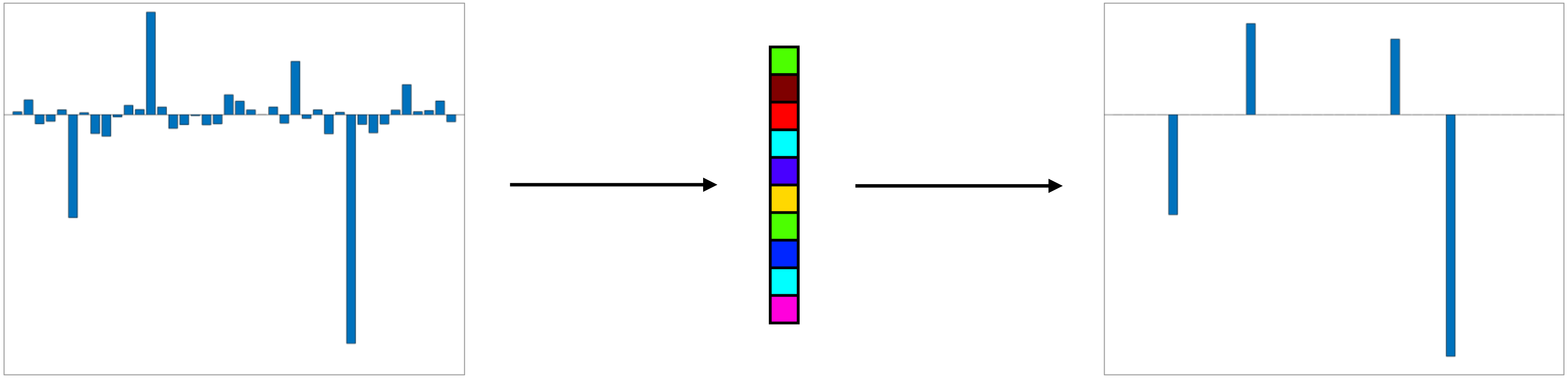
Discrete compressive sensing



$x \in \mathbb{R}^n$ $\qquad$ $y = Mx + w$ $\qquad$ $\tilde{x}$

- **Signal**: vector

Discrete compressive sensing



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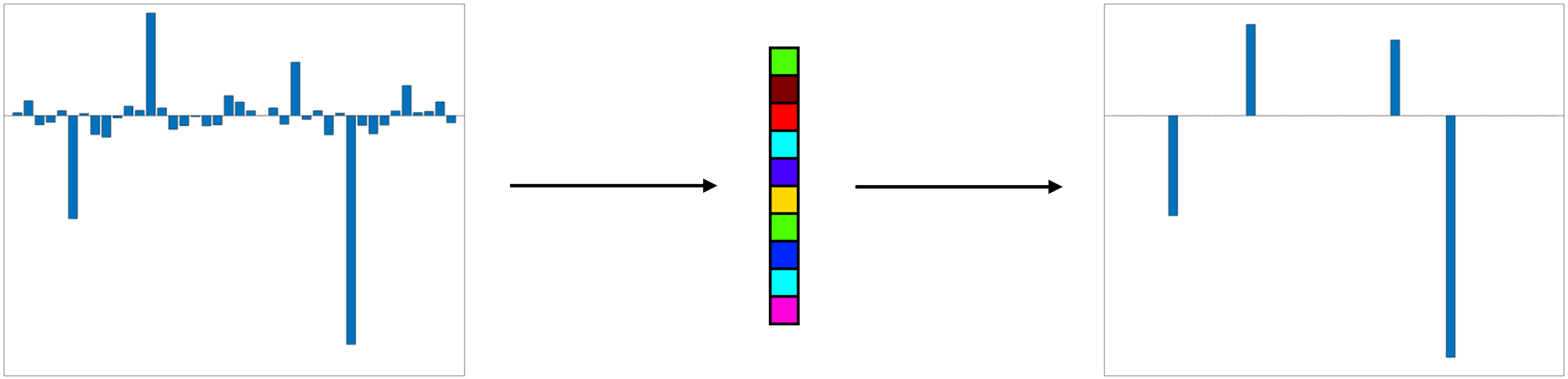
- **Signal**: vector
- **Sparsity**: few non-zeros coefficients

Discrete compressive sensing



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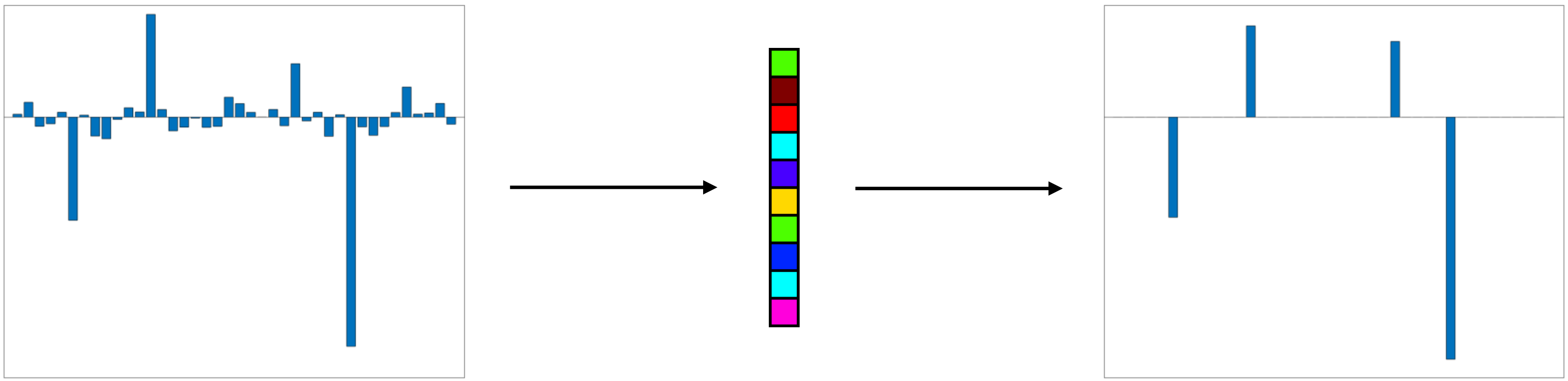

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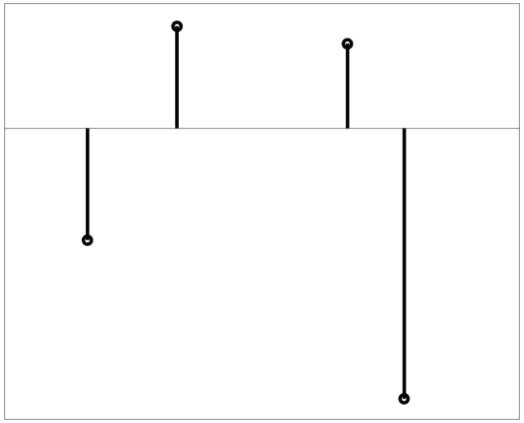
LASSO

$$\min_x \frac{1}{2}\|Mx - y\|_2^2 + \lambda \|x\|_1$$

Continuous sparsity ?



$$\mu \in \mathcal{M}(\mathcal{X})$$

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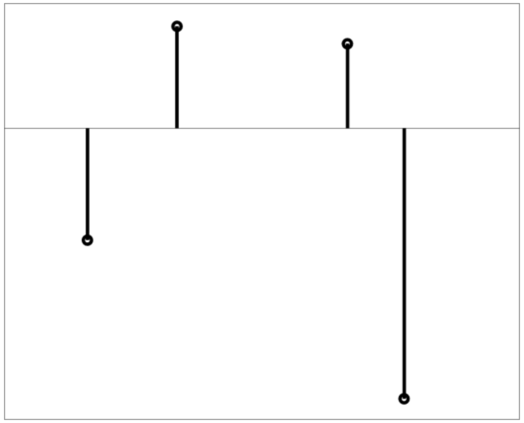

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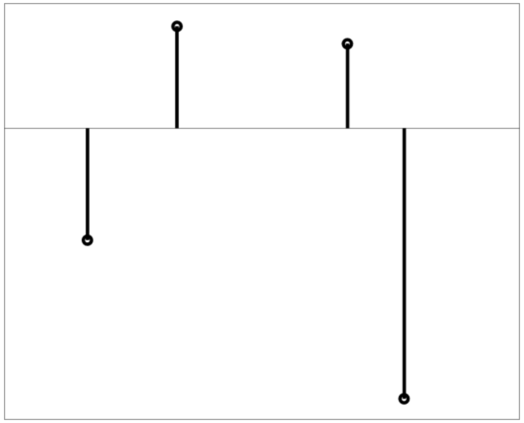

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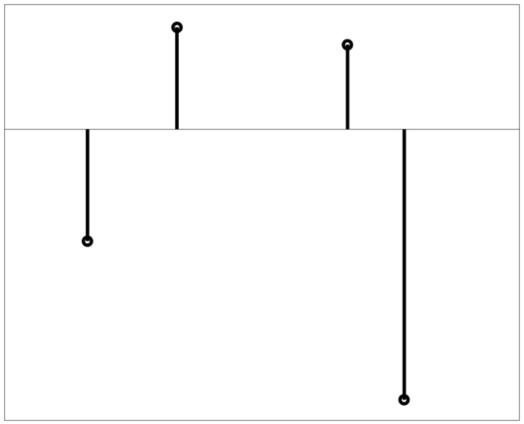

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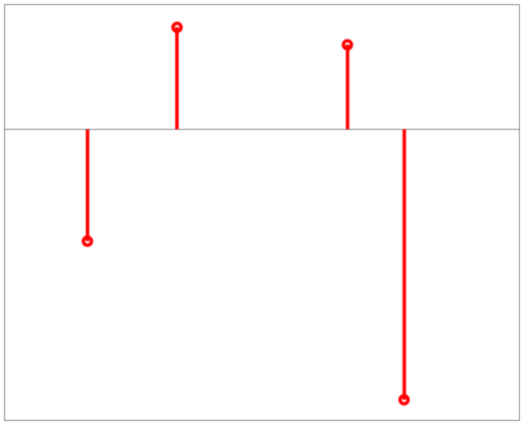

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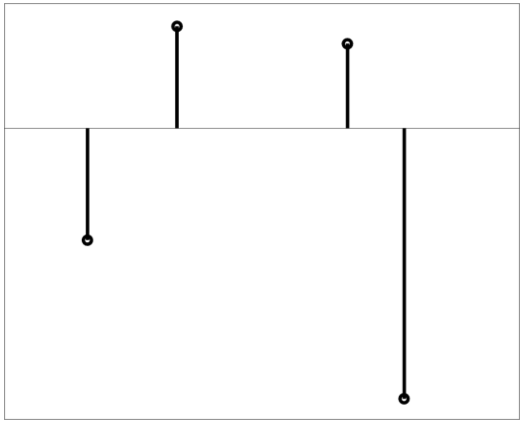

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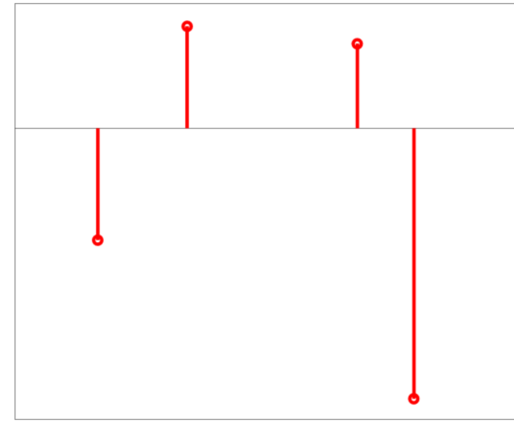

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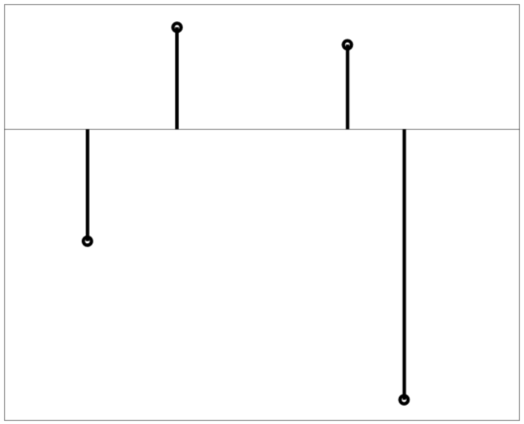

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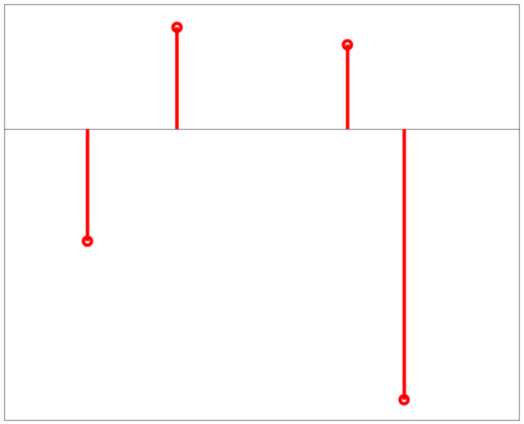

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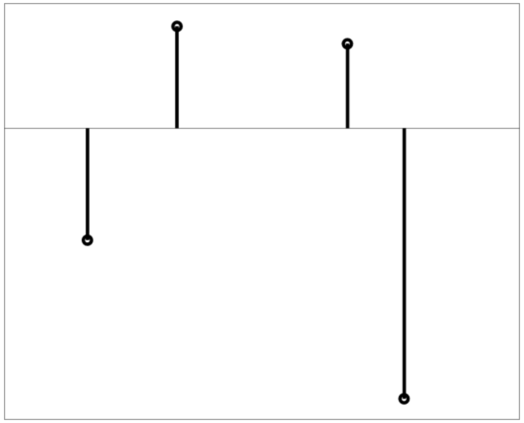

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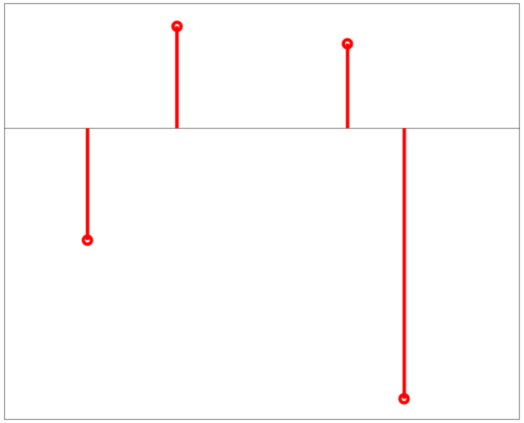

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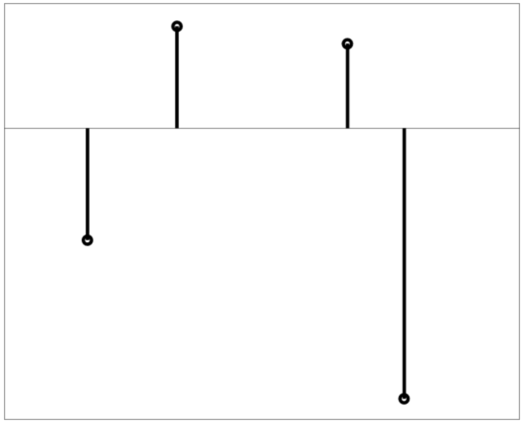

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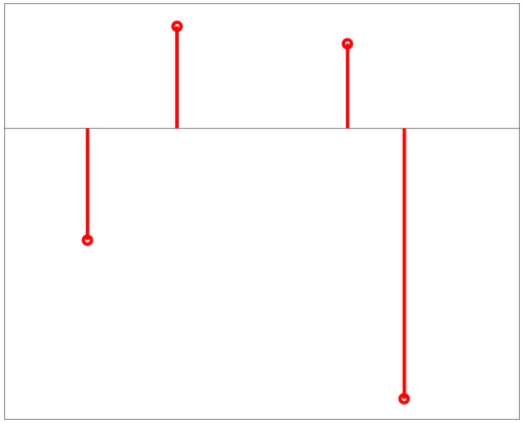

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See Keriven 2017, Gribonval 2017

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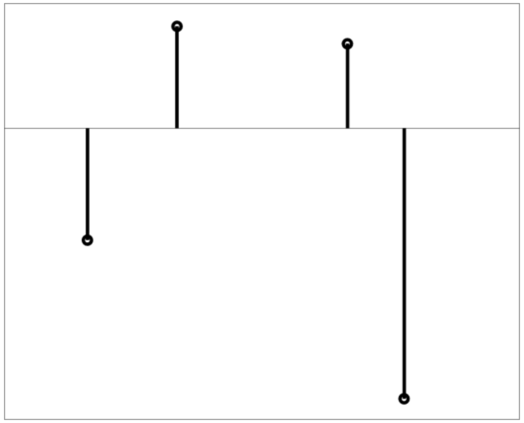

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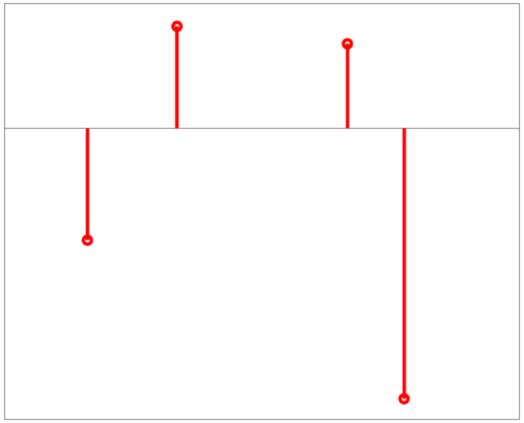

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BLASSO ***[De Castro, Gamboa 2012]***

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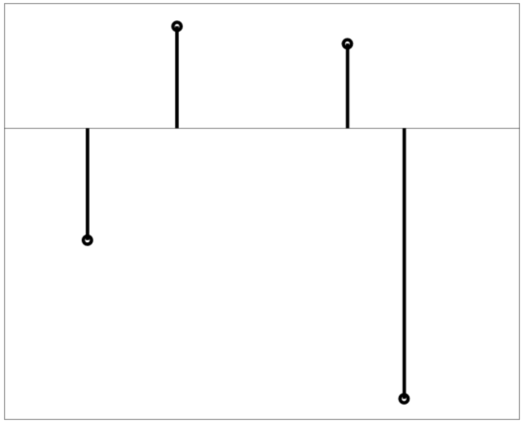

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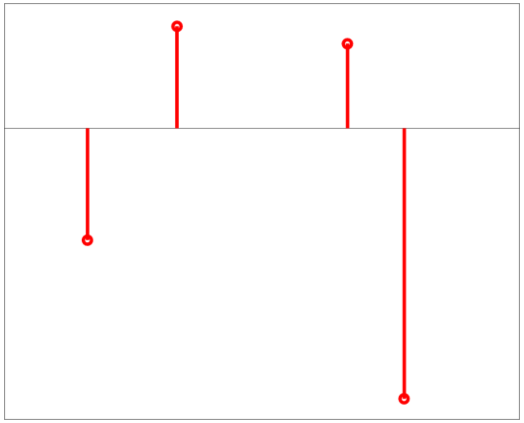

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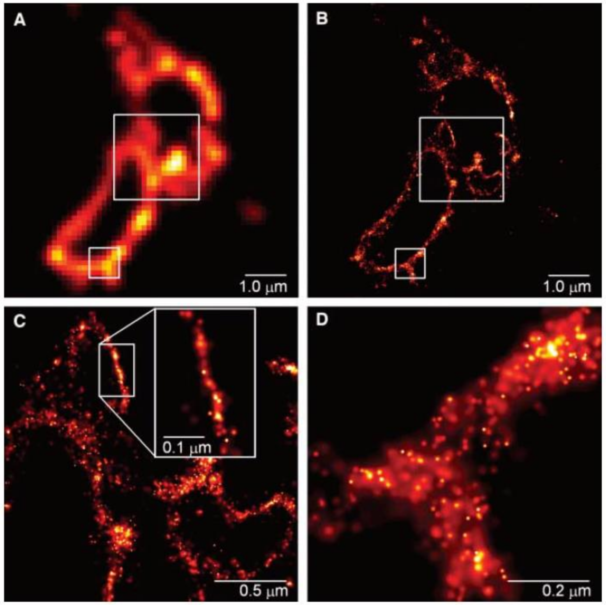
Other approaches: « Prony-like » ESPRIT, MUSIC... (but only 1d noiseless Fourier)

Example of applications



Fluorescence microscopy (3D)
PALM, STORM… *[Betzig 2006]*

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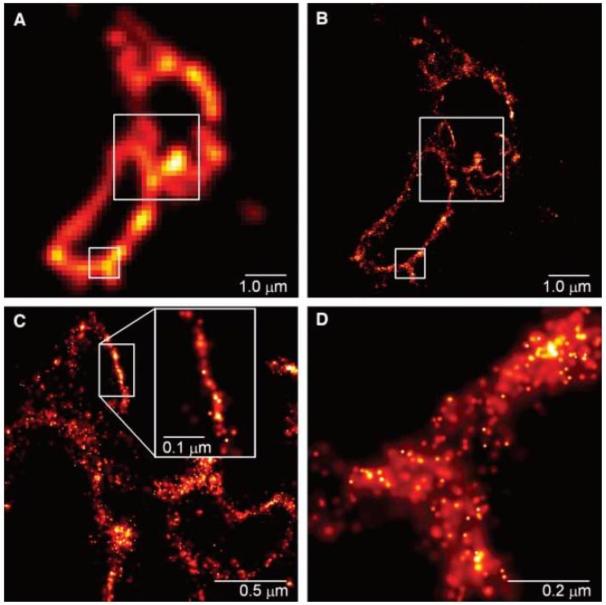

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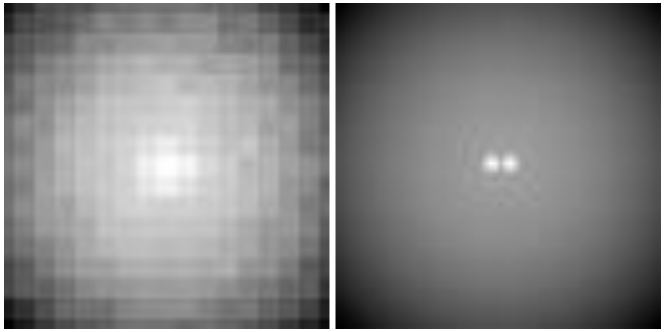

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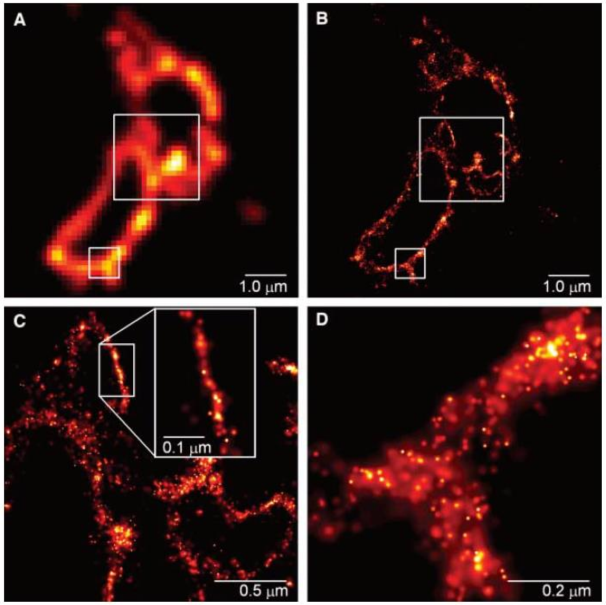

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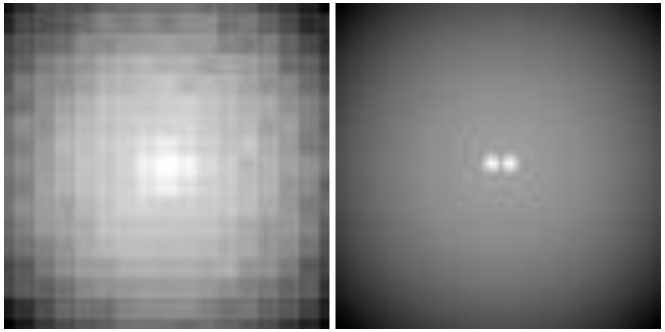
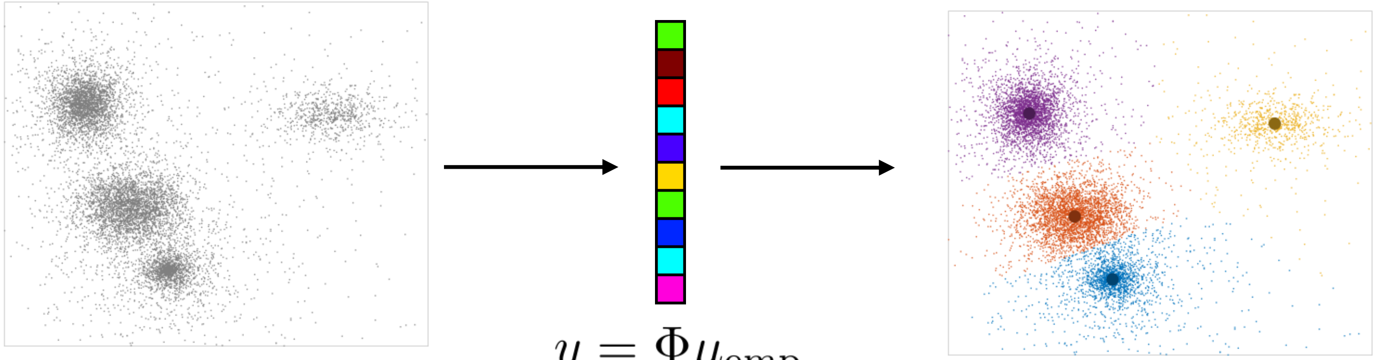


Compressive mixture model learning (many D)
[Keriven 2017]



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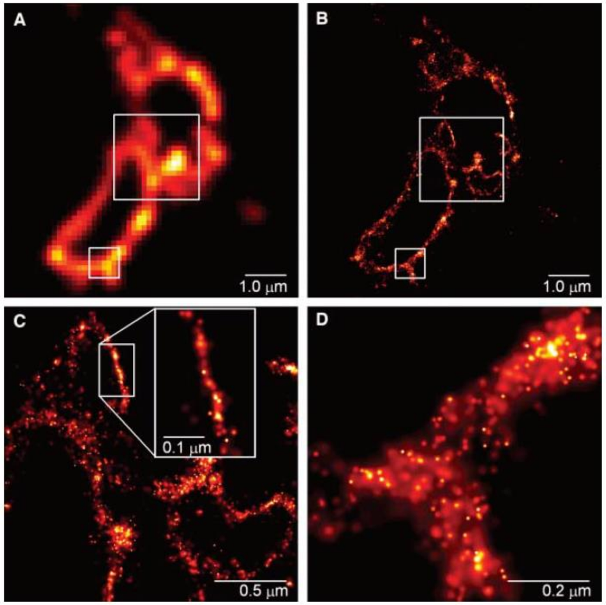

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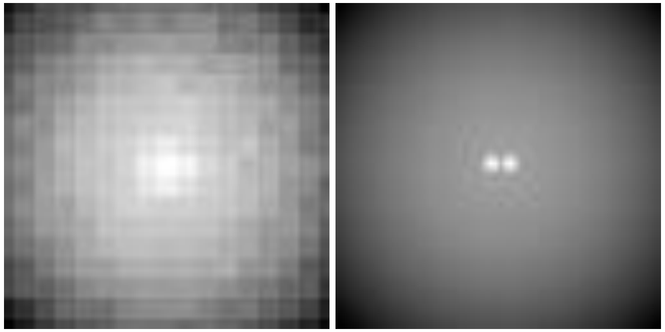

- Neuro-imaging with EEG (3D) *[Gramfort 2013]*
- 1-layer neural network (many D) *[Bach 2017]*
- Radar
- Geophysics
- …

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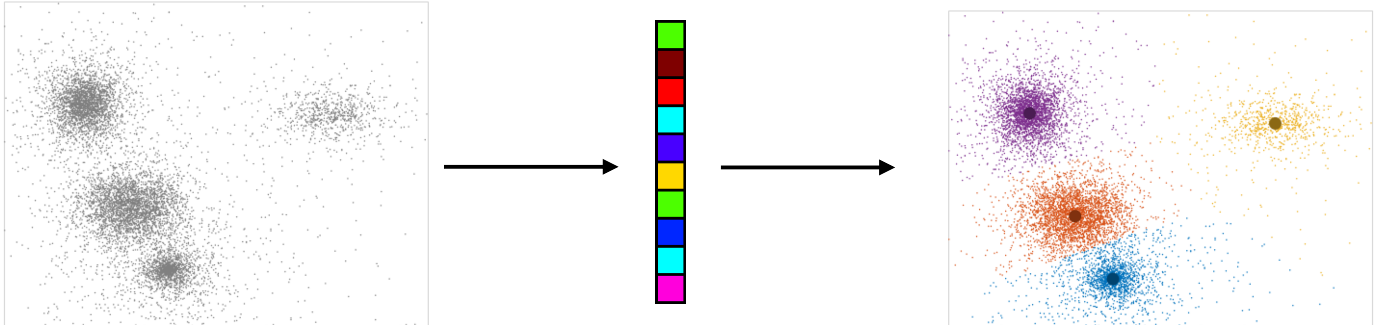

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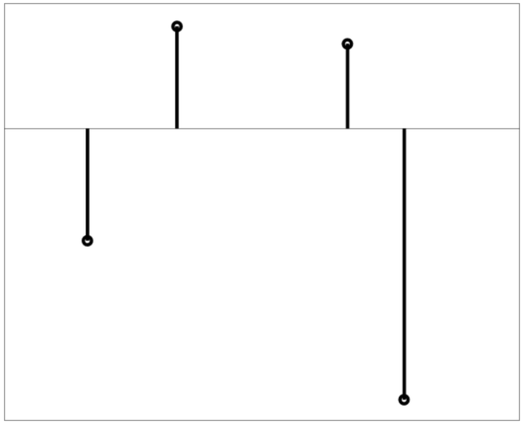

$\mu \in \mathcal{M}(\mathbb{T})$

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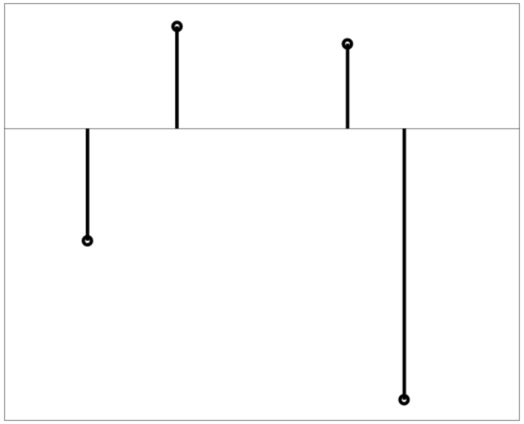

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$\longrightarrow$ **First Fourier coefficients**

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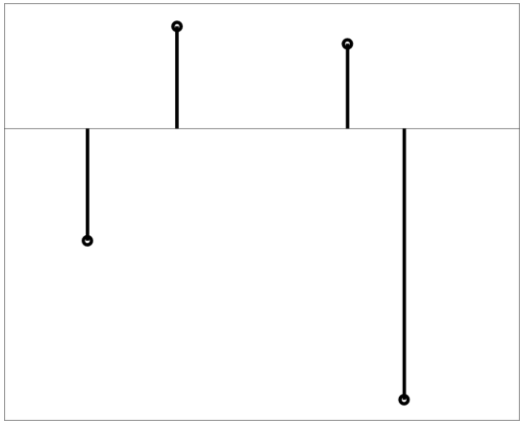

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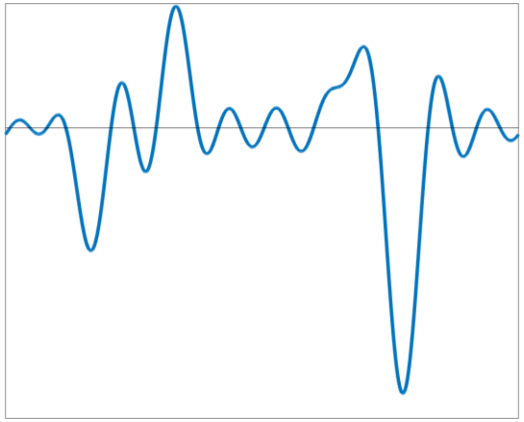
$\xrightarrow{\text{Inverse Fourier}}$



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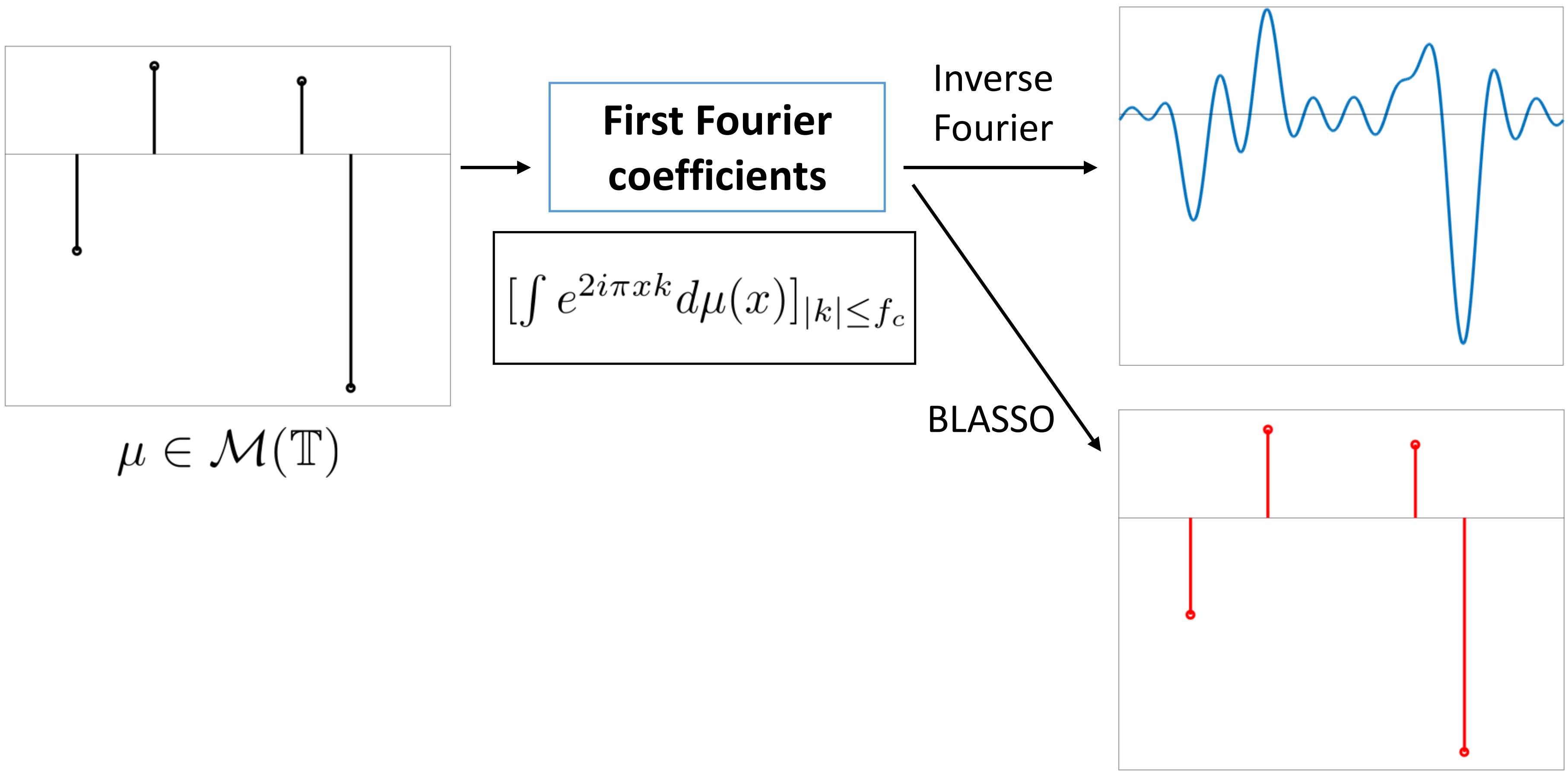

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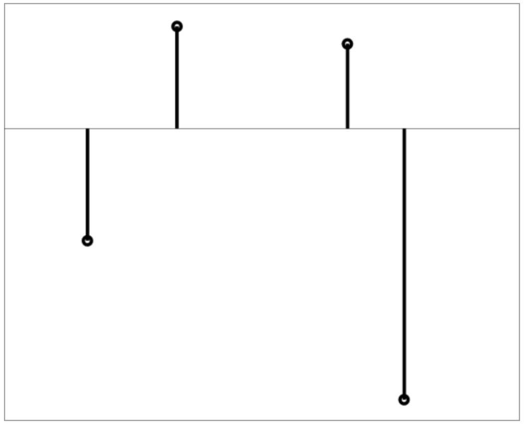

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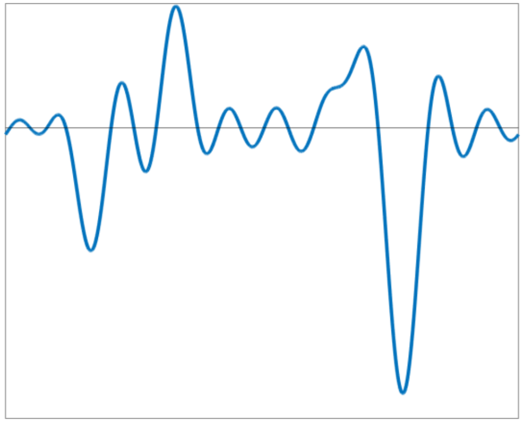
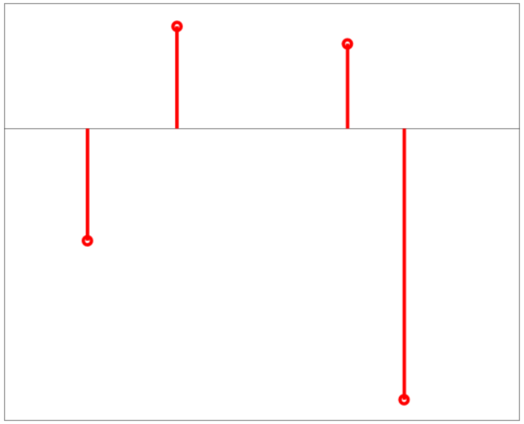


BLASSO



- **Regular Fourier** on the Torus
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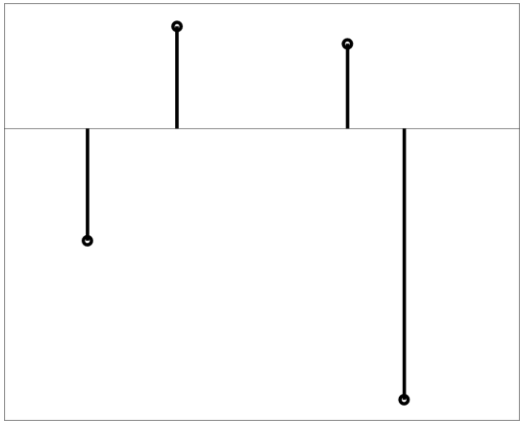

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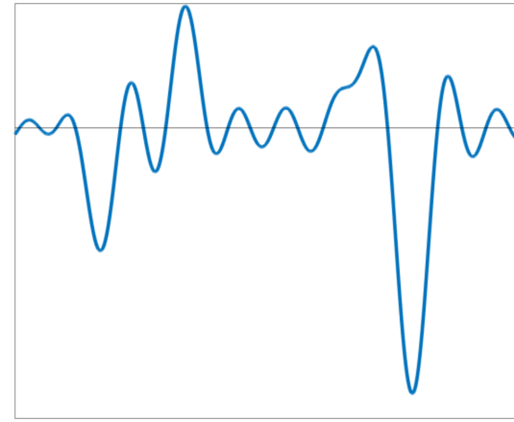

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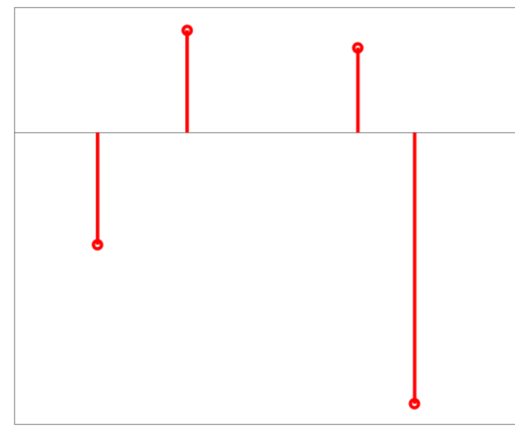

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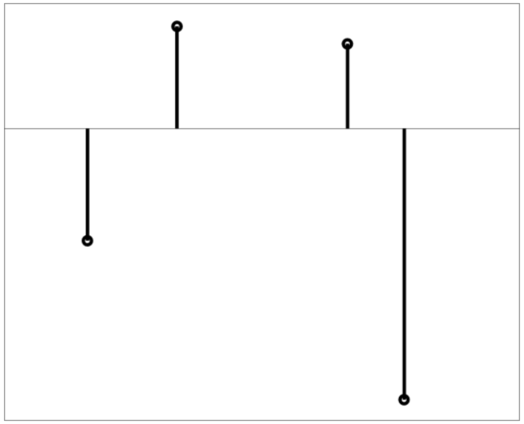

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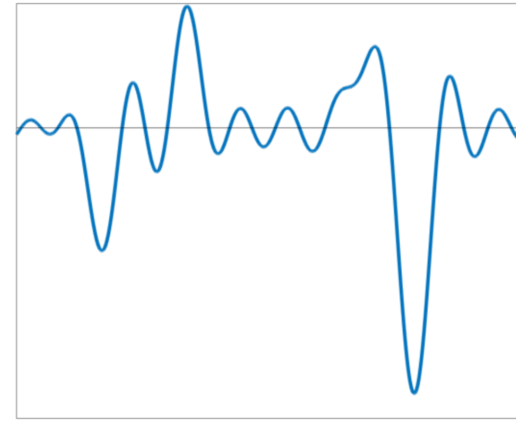

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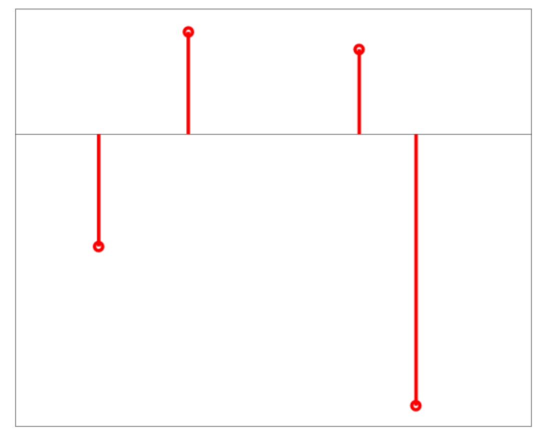

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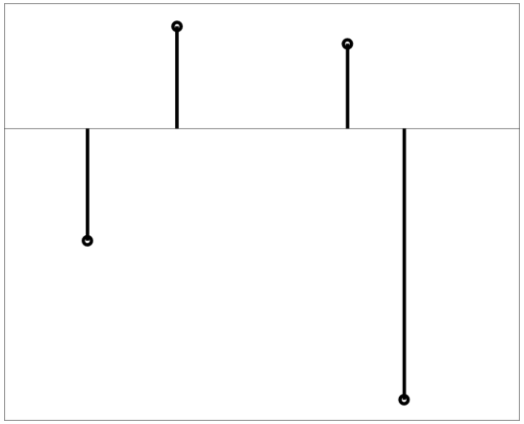

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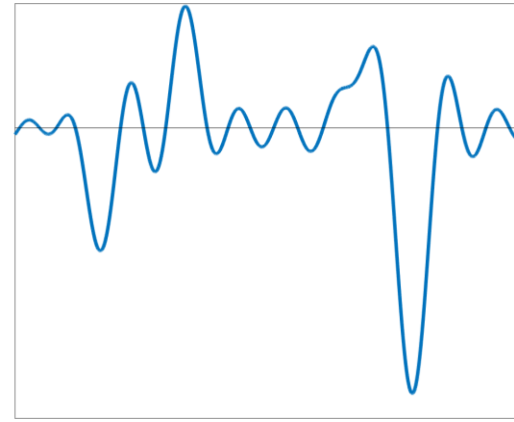

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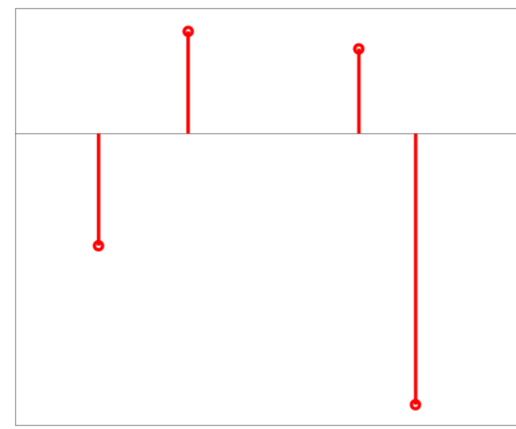

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Many extensions since…

Previous work, contribution

Relevant previous works:

- *[Tang, Recht 2013]*: $m \geq s \log(s) \log(f_c)$ **random** Fourier coefficients are sufficient

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Contributions:

- Generalize to many multi-d measurement operators, express the minimal separation as a **geometry-aware Fisher metric**
- 1: **Remove the random sign assumption** *(weak convergence)*
- 2: Prove **support stability when** $\|w\| \leq s^{-1}$ *(with random signs)*

Outline

1. Background on dual certificates
2. Minimal separation and Fisher metric
3. Main results, applications
4. Conclusion, outlooks

Dual certificates

Random linear operator:

$$\omega_1, \dots, \omega_m \overset{iid}{\sim} \Lambda$$

$$\Phi\mu = \frac{1}{\sqrt{m}} \left[\int \varphi_{\omega_k}(x) d\mu(x) \right]_{k=1}^{m}$$

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μ_0 solution of BLASSO $\Leftrightarrow$ $\frac{1}{\lambda}\Phi^\star(\Phi\mu_0 - y) \in \partial|\mu_0|(\mathcal{X})$

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Dual certificate **(noiseless case)**

μ_0 solution of $\displaystyle\min_{\Phi\mu = y} |\mu|(\mathcal{X})$ $\Leftrightarrow$ $\mathrm{Im}(\Phi^\star) \cap \partial|\mu_0|(\mathcal{X}) \neq \emptyset$

What does it look like ?

What is a dual certificate ?

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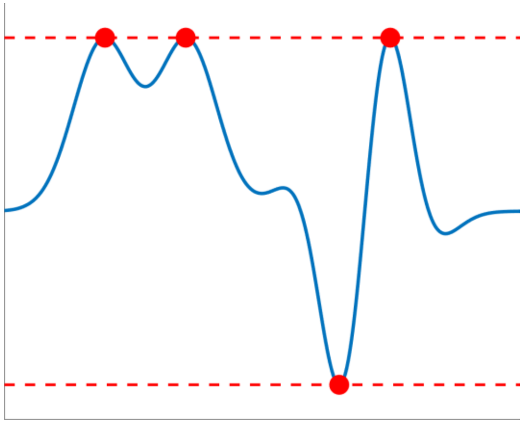
$$\boxed{\begin{gathered}\eta(x_i) = \mathrm{sign}(a_i) \\ \|\eta\|_\infty \leq 1\end{gathered}}$$



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Non-degenerate dual certif.

$$|\eta(x)| < 1$$

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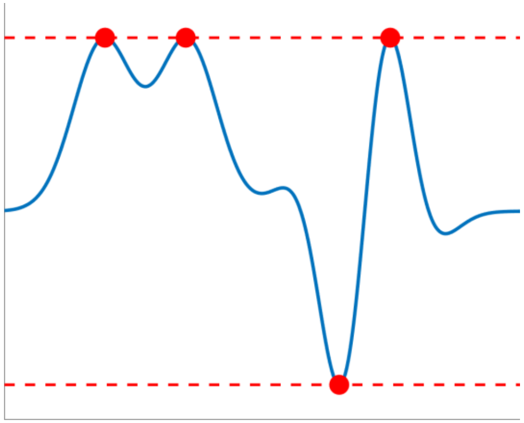

Ensures **uniqueness** and **robustness**...

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Intuitively:

Larger m -> easier interpolation

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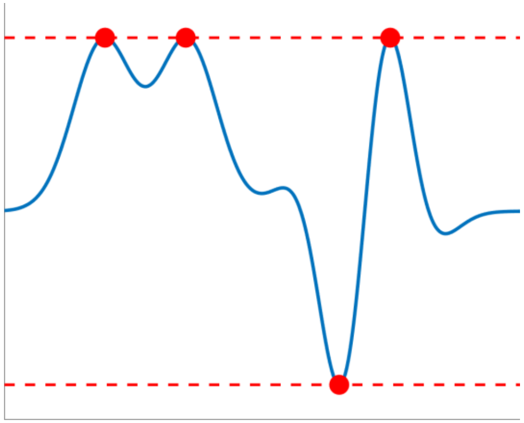

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Proof strategy

Step 1: Study the limit case $m \to \infty$ to derive an appropriate notion of minimal separation

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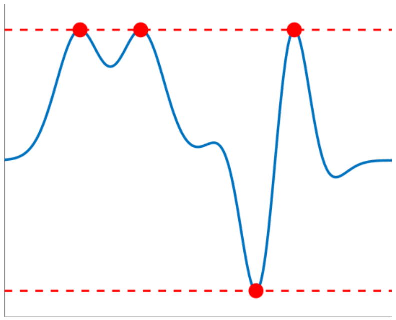

$m \to \infty$

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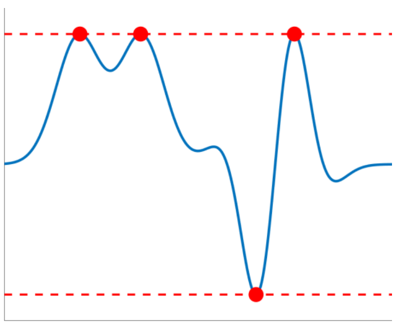

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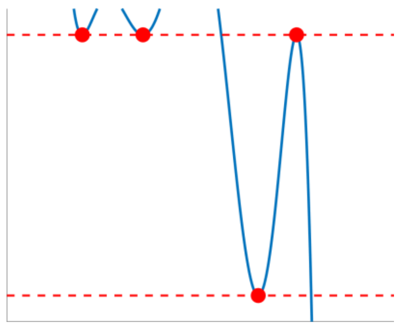

m=10

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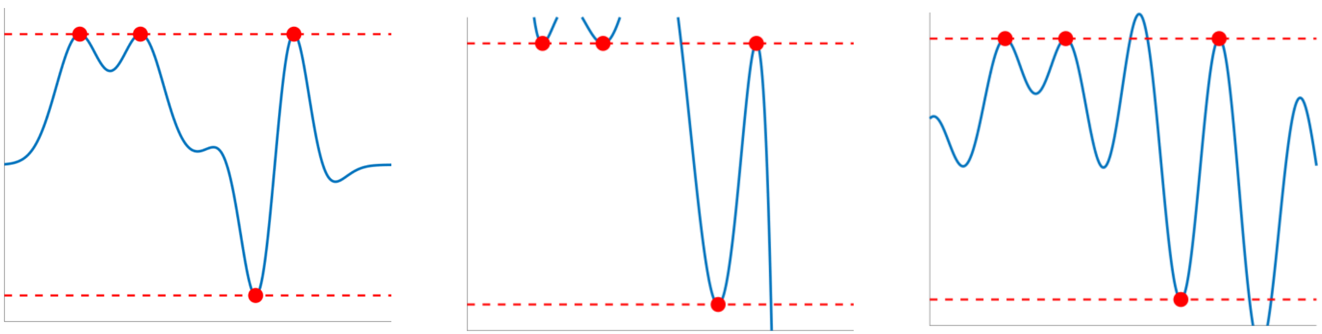

$m \to \infty$ *m=10* *m=50*

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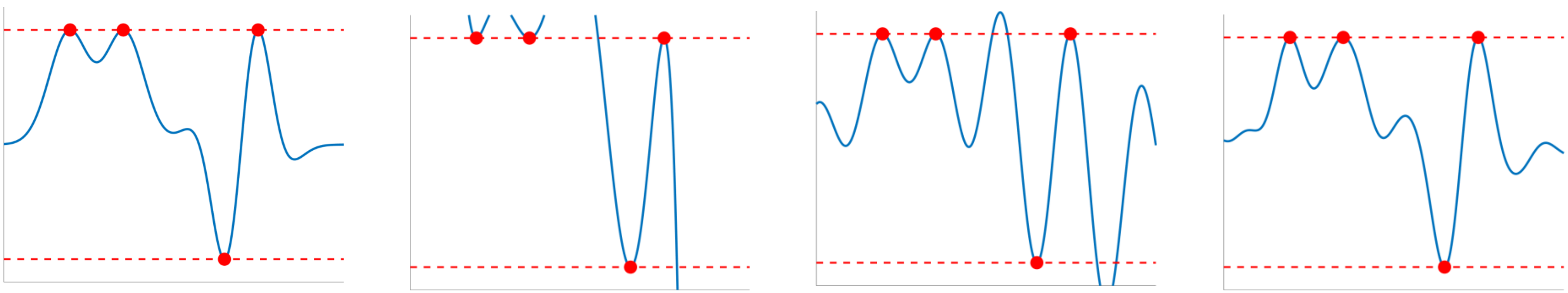

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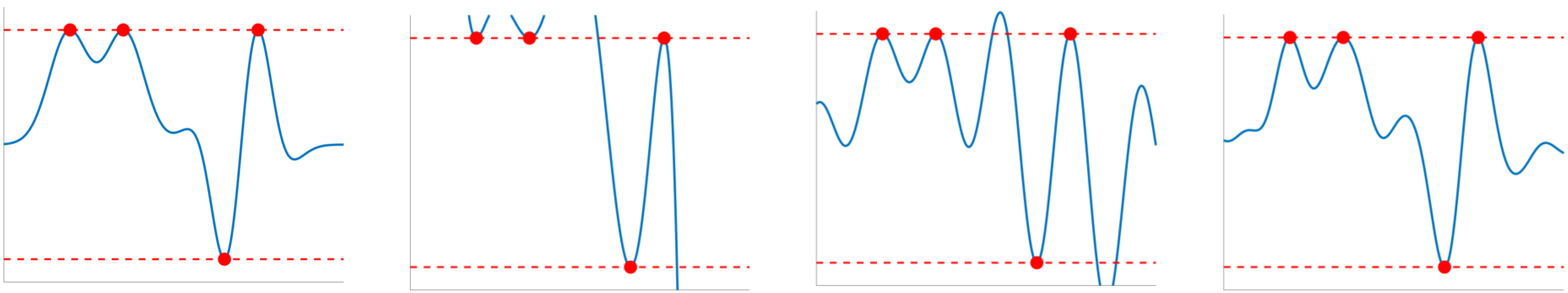

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Step 3: recovery

- Adaptation of *[Azaïs 2015]* for weak convergence
- Quantitative Implicit Function Theorem *[Denoyelle 2015]* for support stability

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How to construct a certificate ?

Study **limit covariance kernel** when $m \to \infty$:

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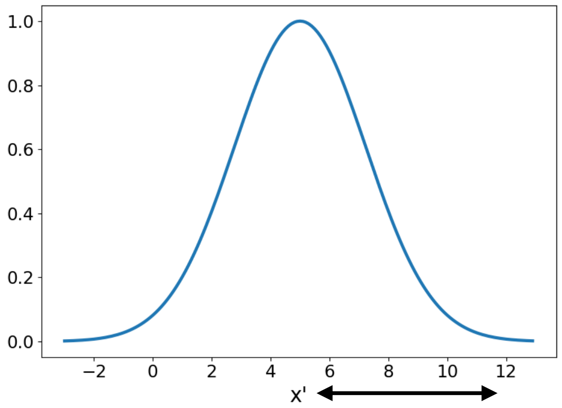

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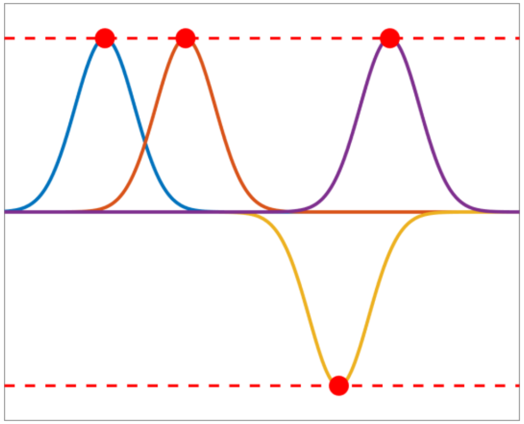

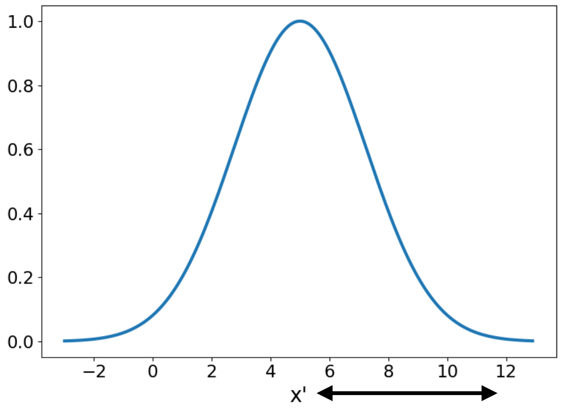

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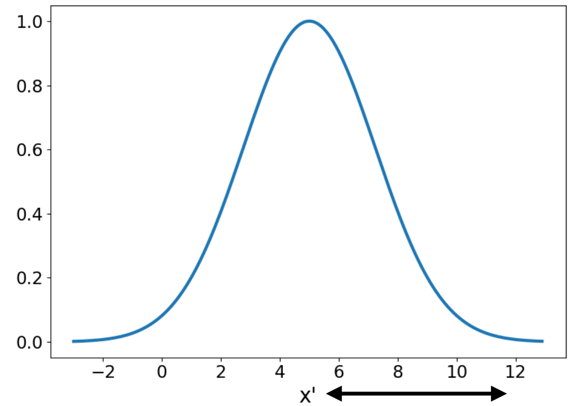

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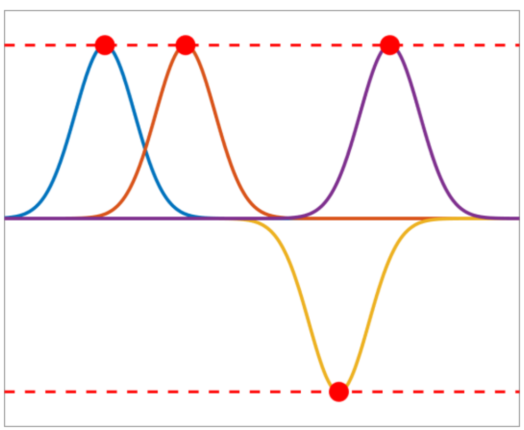

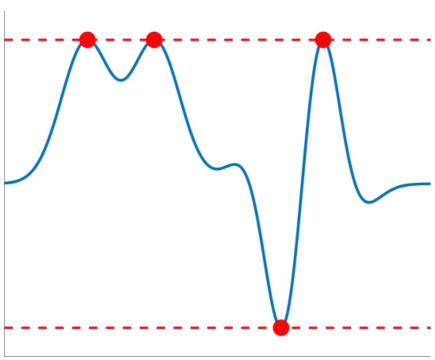

2: ***Small*** adjustements (minimal separation)

Minimal separation and Fisher metric

Which metric for separation ?

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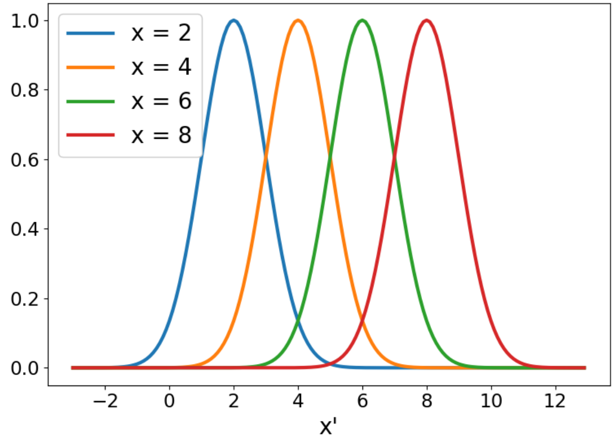
Classical case: translation-invariant kernel



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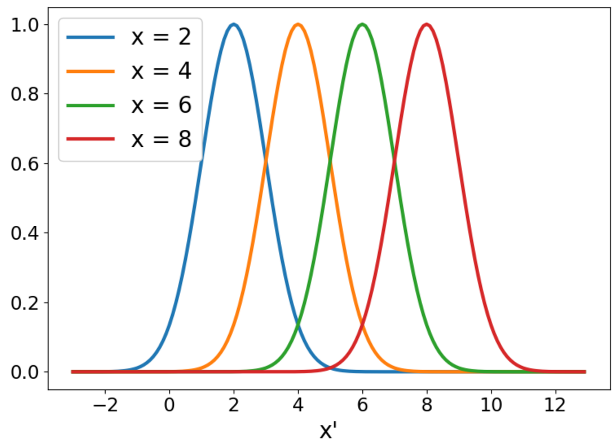

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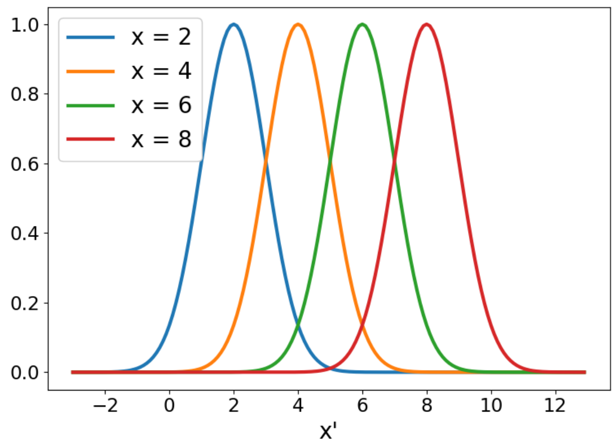

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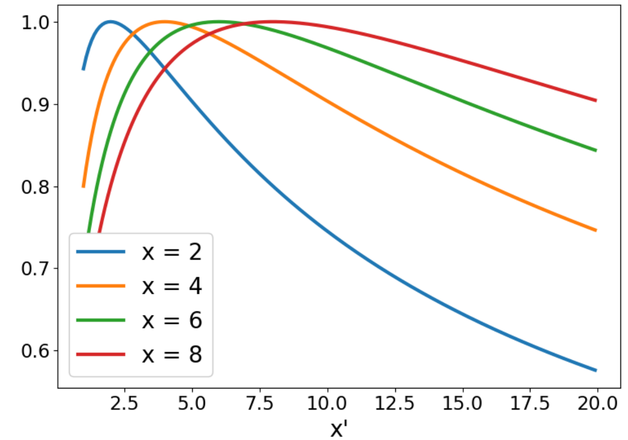

Kernel for microscopy

Minimal separation and Fisher metric

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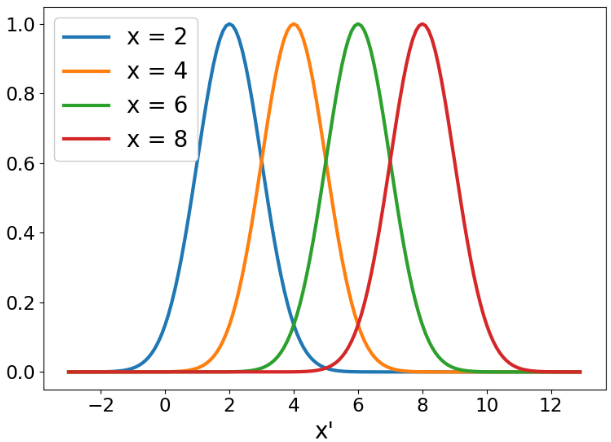

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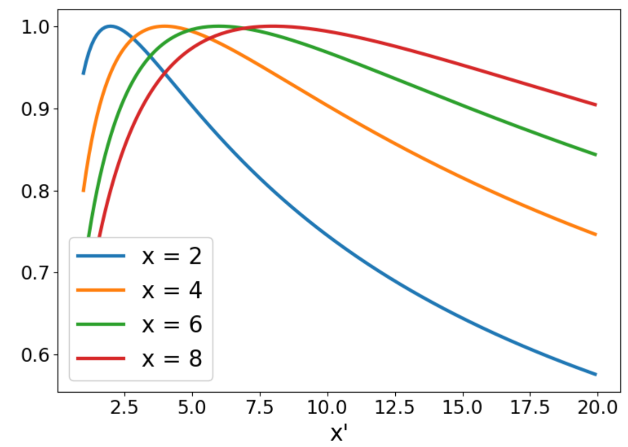

Kernel for microscopy

Riemannian metric associated to a kernel *[Amari 99]*:

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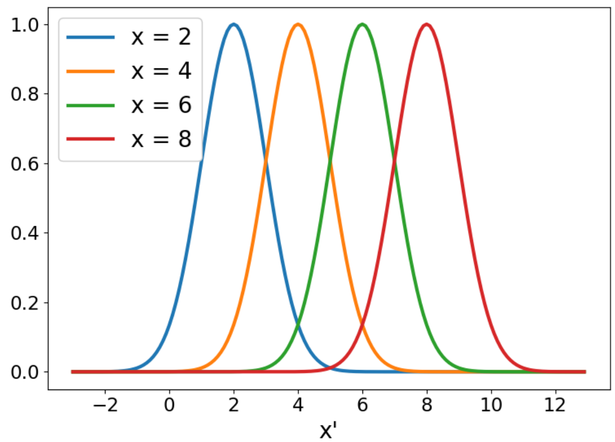

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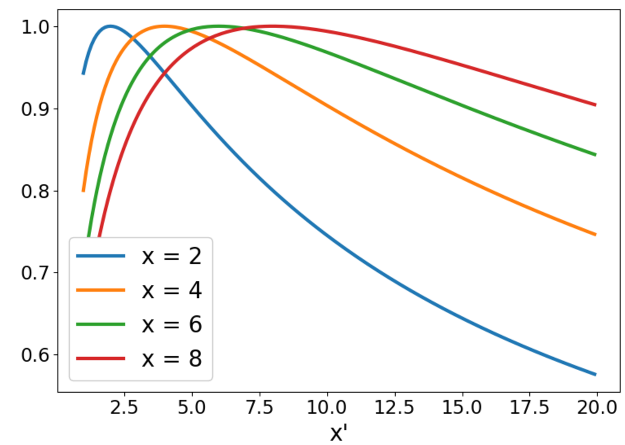

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Thm: under some hypothesis, for $d_H(x_i,x_j) \geq \Delta$, there exists non-degenerate η

Examples

Kernel

Features

Fisher metric and minimal separation

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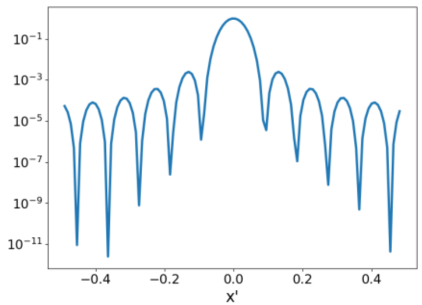
Kernel

Discrete Fourier on Torus:

Féjer kernel



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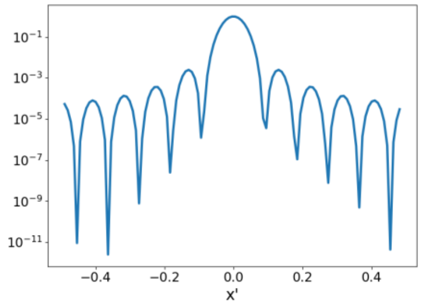

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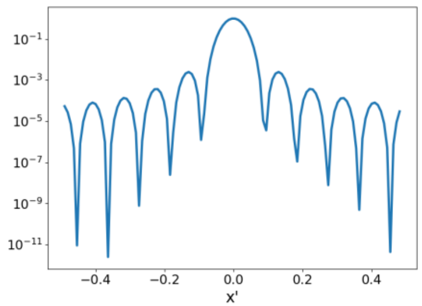

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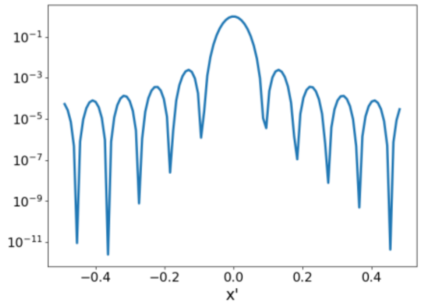
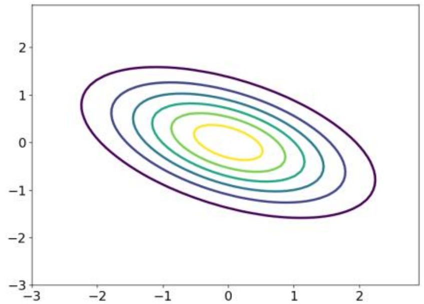


Continuous Gaussian Fourier:
Gaussian kernel



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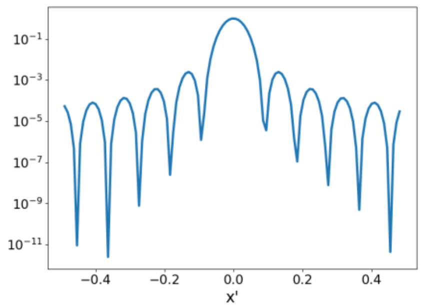

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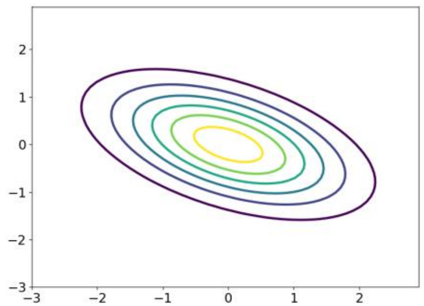

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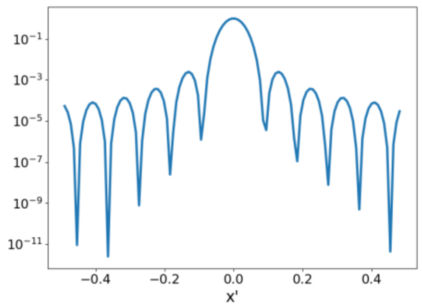
Féjer kernel



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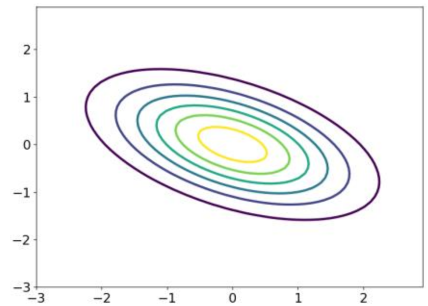

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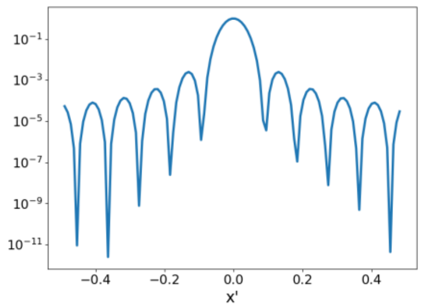

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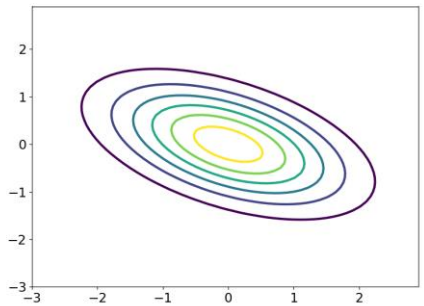

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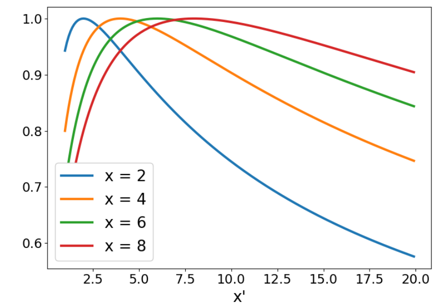

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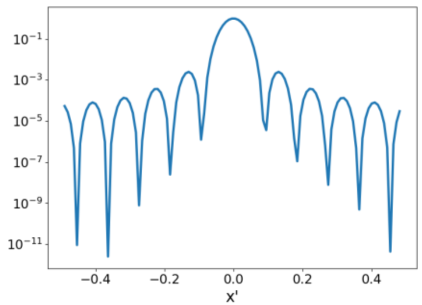

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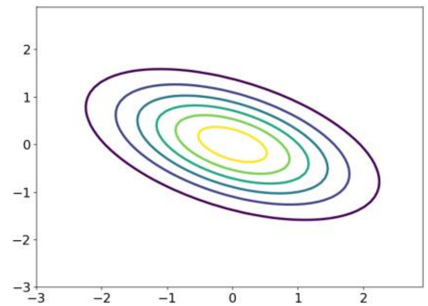

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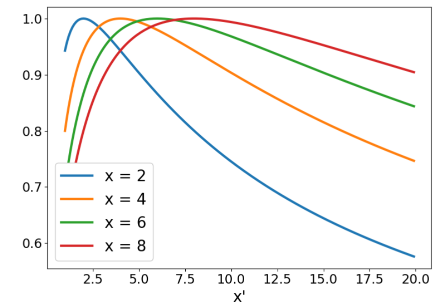

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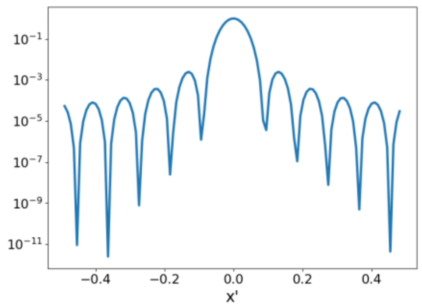

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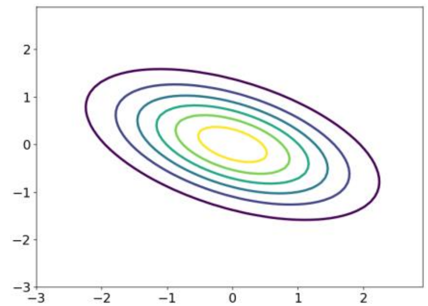

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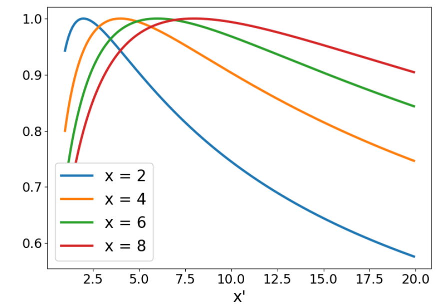

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1. Background on dual certificates
2. Minimal separation and Fisher metric
3. Main results, applications
4. Conclusion, outlooks

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- The recovered measure is formed of exactly s Diracs

$$\sqrt{\sum_i |\tilde{a}_i - a_i|^2 + d_H(\tilde{x}_i, x_i)^2} \lesssim \frac{\sqrt{s}}{\min_i |a_i|}(\|w\| + \lambda)$$

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Then: if $m \geq s^{3/2} d^3, \quad n \geq s^2 d^6, \quad \|x_i - x_j\|_{\Sigma^{-1}} \geq \sqrt{d \log(s)}$

The BLASSO yields exactly s Diracs: *non-asymptotic* model selection !

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- **Outlooks**
  - **Implication of support stability for algorithms ?** (active field)
  - Better characterization of the « universality » of the geodesic distance
  - More quantified treatment of dimension
  - Other practical applications (eg 1-layer neural networks with continuum of neurons *[Bach 2017]*)

Thank you !

Poon, Keriven, Peyré. **A Dual Certificates Analysis of Compressive Off-the-Grid Recovery.** *Preprint arxiv:1802.08464*

Poon, Keriven, Peyré. **Support Localization and the Fisher Metric for off-the-grid Sparse Regularization.** *Preprint arxiv:1810.03340*

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