Sparse and Smooth:
Spectral Clustering in the dynamic SBM

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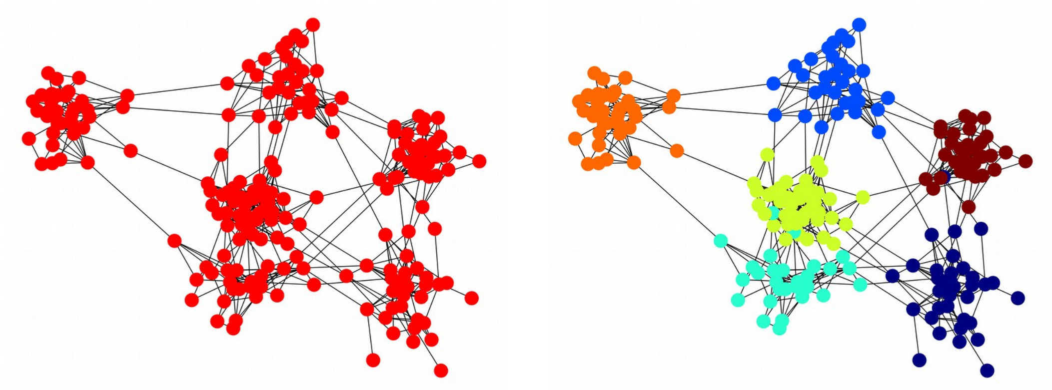
Spectral Clustering



Cluster the nodes of a graph using its structure.

Application in:
- Social network analysis
- Point cloud segmentation
- Etc, etc

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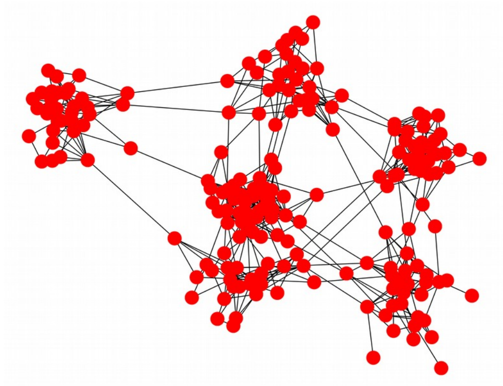

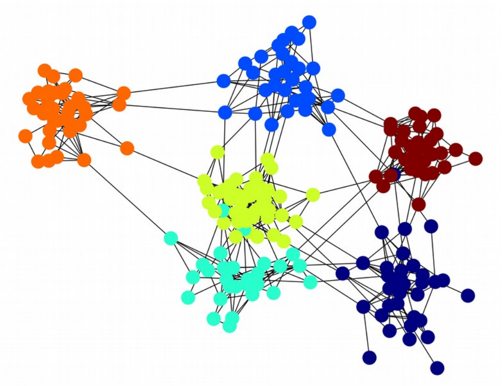

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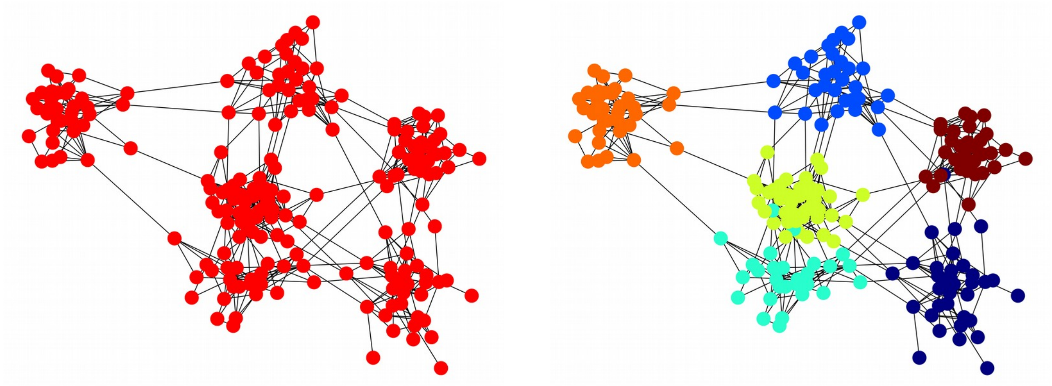

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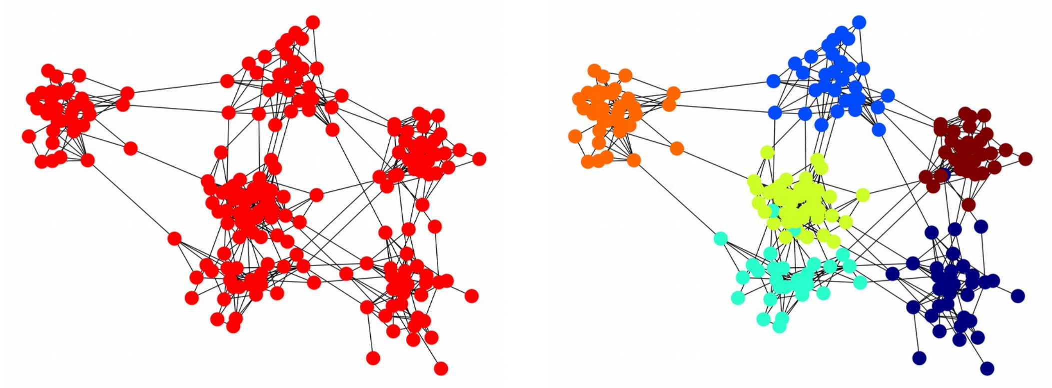

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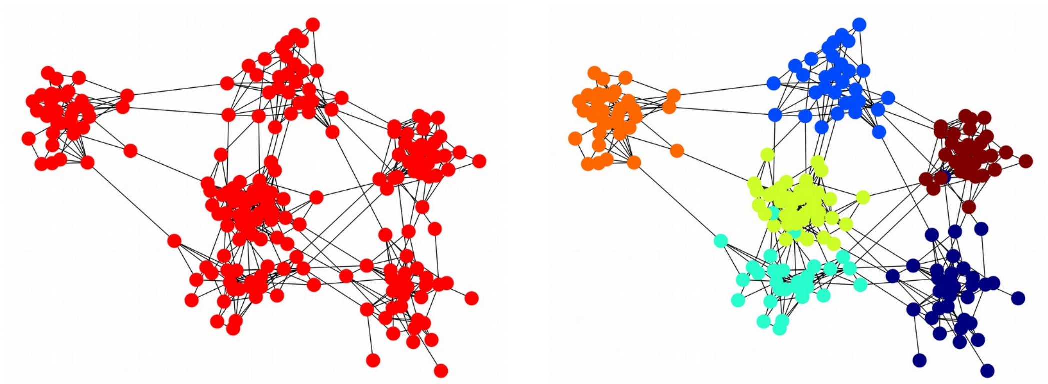

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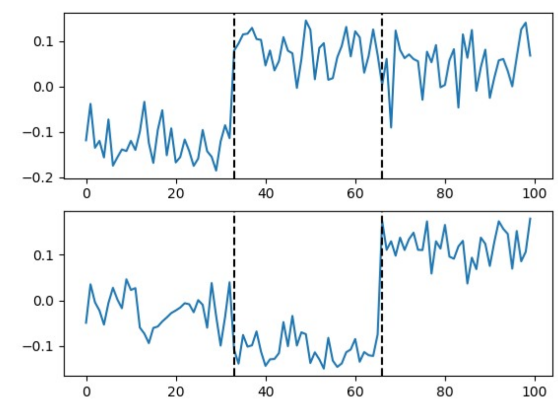
- Cluster the rows of $\tilde{U}$ with k-means

Eigenvectors :



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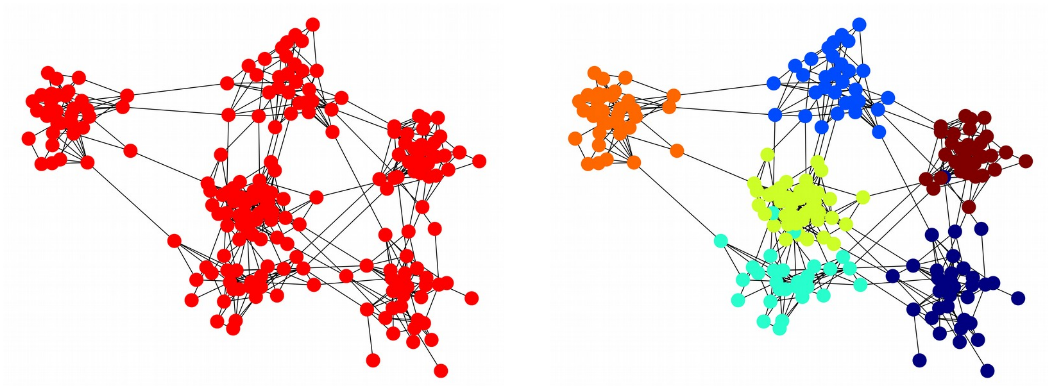

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- Many many (fast) variants...

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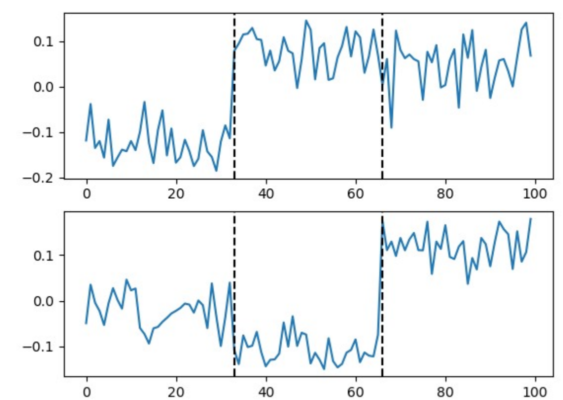
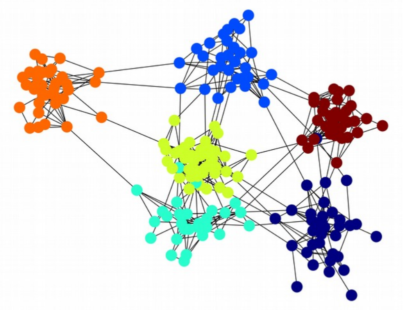


SC and SBM

Stochastic Block Model (SBM)



$\Theta \in \{0,1\}^{n \times K}$ $\Theta_{ik} = 1$ *if node* i *belongs to comm.* k

$B \in [0,1]^{K \times K}$ *Symmetric probability matrix*

$$\begin{cases} a_{ij} \sim \mathrm{Ber}(B_{k\ell}) \\ \text{when } \Theta_{ik} = \Theta_{j\ell} = 1 \end{cases}$$

Independent edges

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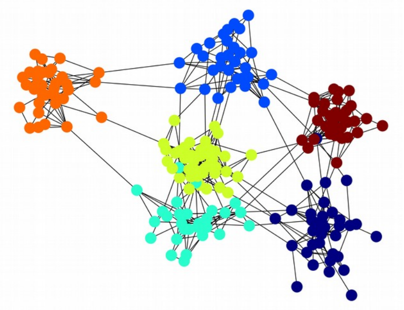

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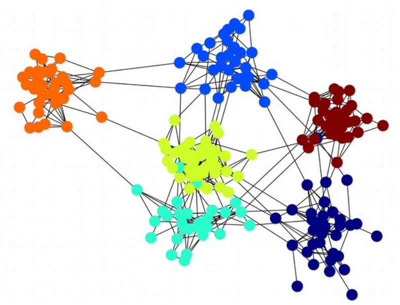

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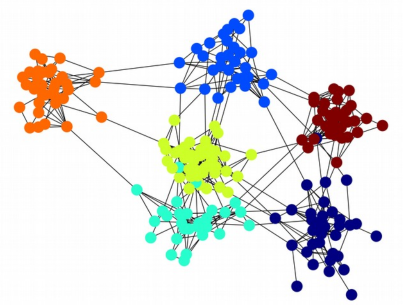

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  - Easy
- **Sparse** $\alpha_n \sim 1/n$
  - Hard *(some **asymptotic** results from statistical physics)* *[Krzakala, Mossel, Massoulié, Abbe...]*
- ***Relatively* sparse** $\alpha_n \sim \log n / n$
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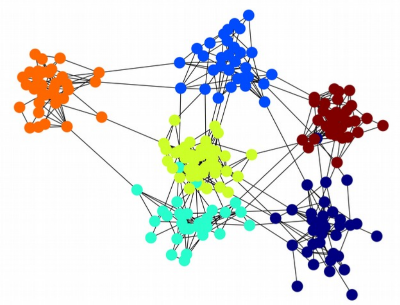

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- SC with $W = A$
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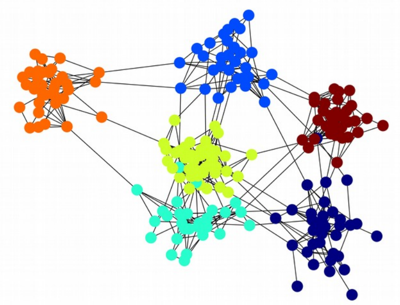

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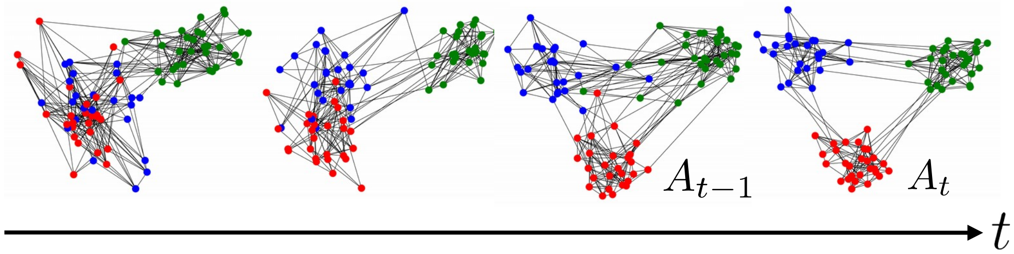
- With proba $1-n^{-r}$:

$$L(\hat\Theta,\Theta) \leq C_r \frac{K^2}{n\alpha_n(1-\tau)^2}$$

$$L(\hat\Theta,\Theta) = n^{-1}\min_\sigma \|\hat\Theta\sigma - \Theta\|_0$$

SC and DSBM



Goal: Exploit past data to

- Track communities
- Enforce smoothness/consistency
- **Improve result at time t**

Many approaches (Bayesian, variational…)

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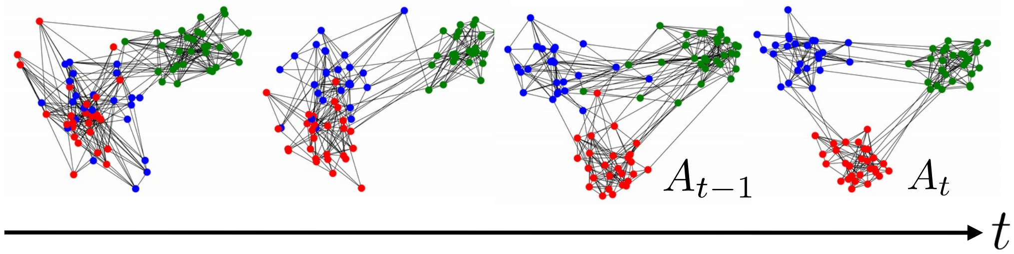

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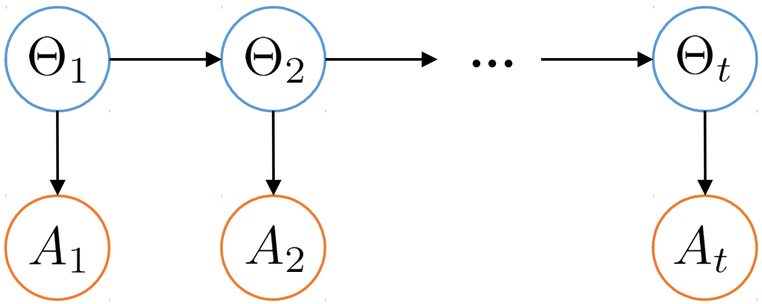

HMM
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- To simplify, B does not change

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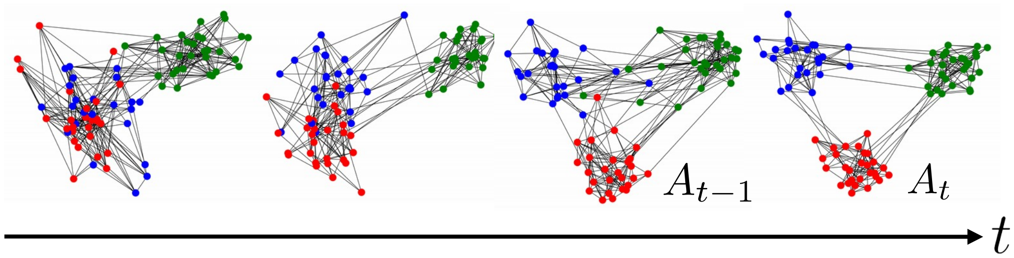

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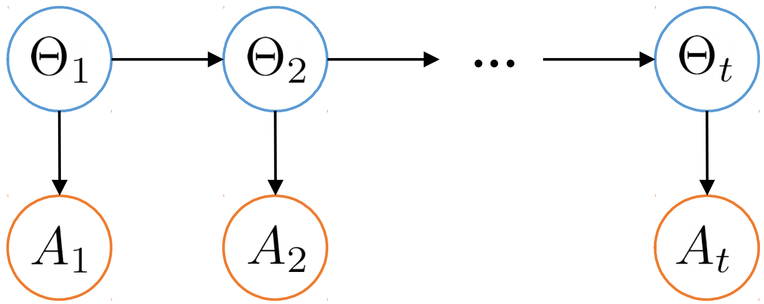

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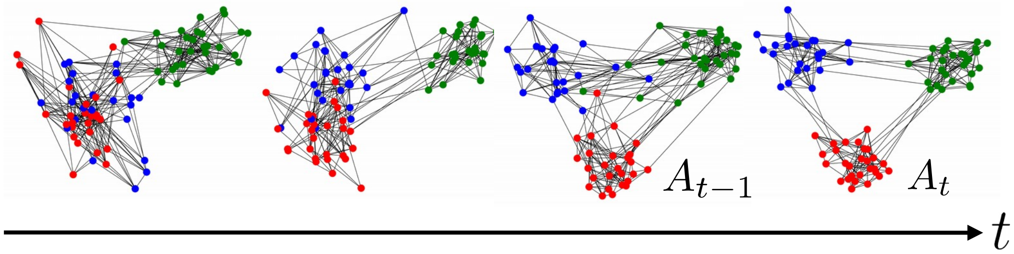

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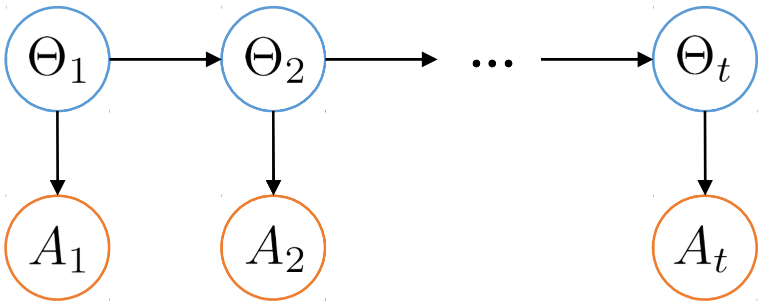

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- Choose $r \sim \rho_n^{-1}$ with

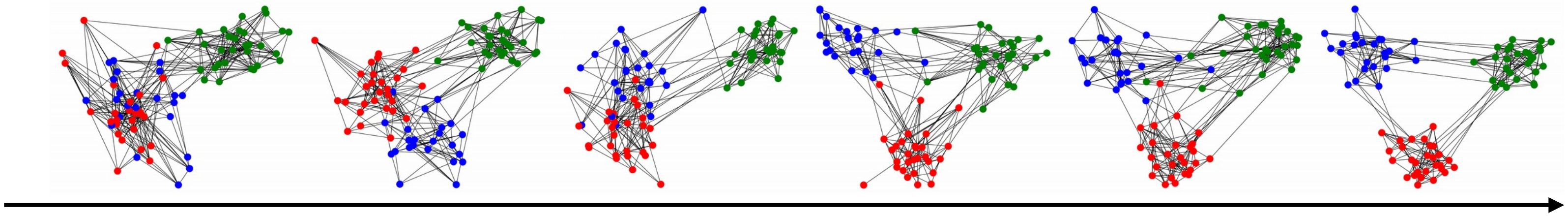
$$\rho_n = \min(1, \sqrt{\alpha_n n \varepsilon_n})$$

- With proba $1 - n^{-r}$:

$$L(\hat{\Theta}, \Theta) \lesssim C_r \frac{K^2}{n\alpha_n(1-\tau)^2}\rho_n$$

Better if ***smooth enough*** $\varepsilon_n = o(1/(\alpha_n n))$

Sparse and smooth...?



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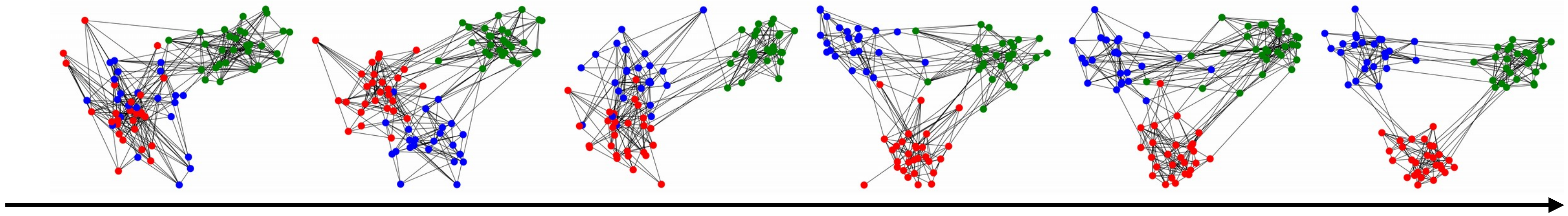

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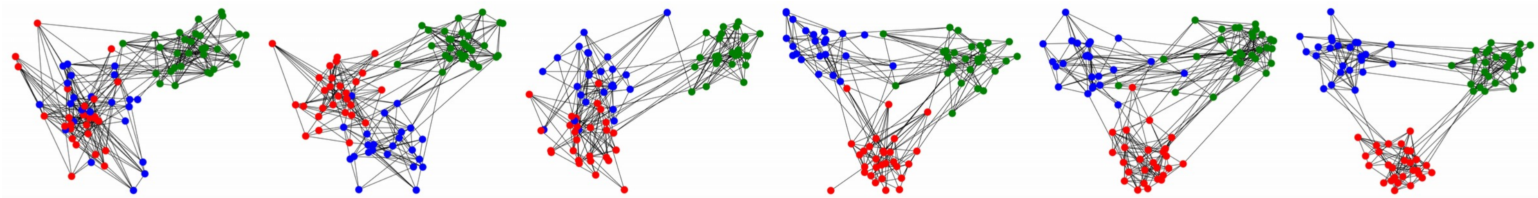

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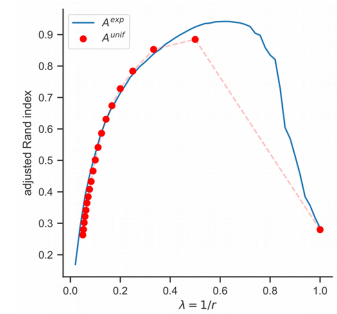
Exponential performs better in practice, and λ *is easier to choose than* r



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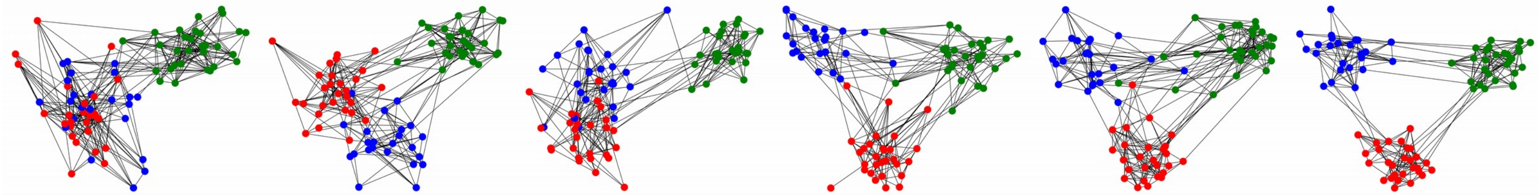

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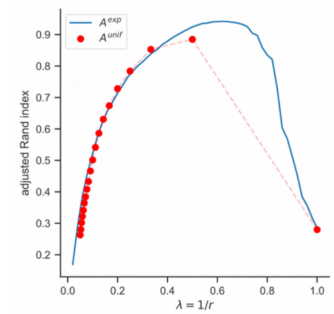

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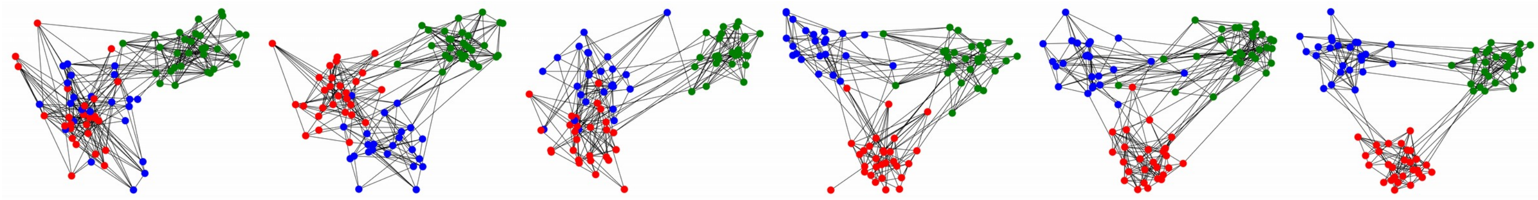

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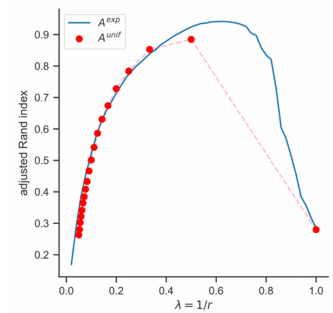

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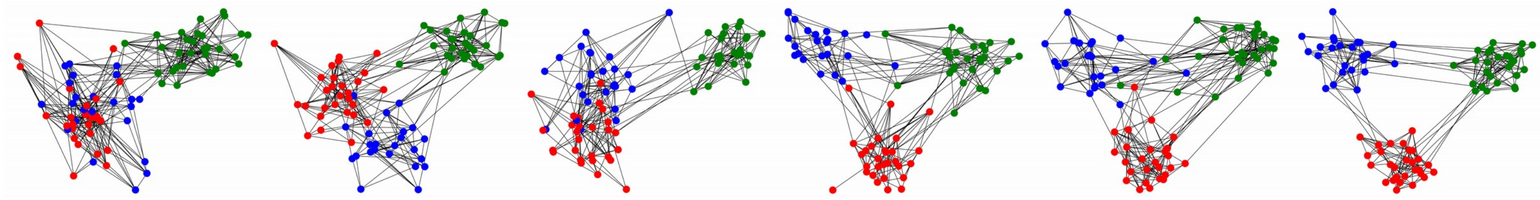

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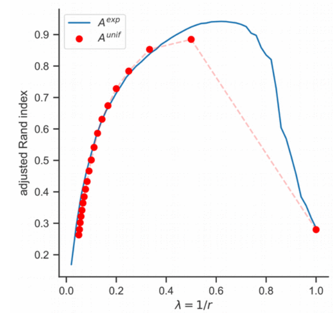

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In practice, the *normalized Laplacian* works better. $L(A) = D(A)^{-1/2} A D(A)^{-1/2}$

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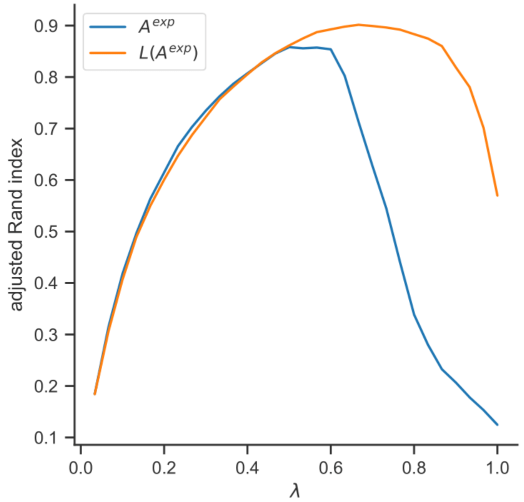
- SC with $W = L\left(\sum_k \beta_k A_{t-k}\right)$
- Sparsity: $\frac{\alpha_n}{\rho_n} \geq C'_r \frac{\log n}{n}$
- With proba $1 - n^{-r}$:

$$L(\hat\Theta, \Theta) \lesssim C_r \frac{K^2}{n\alpha_n(1-\tau)^2} \rho_n$$

The multiplicative constant is lower-bounded !

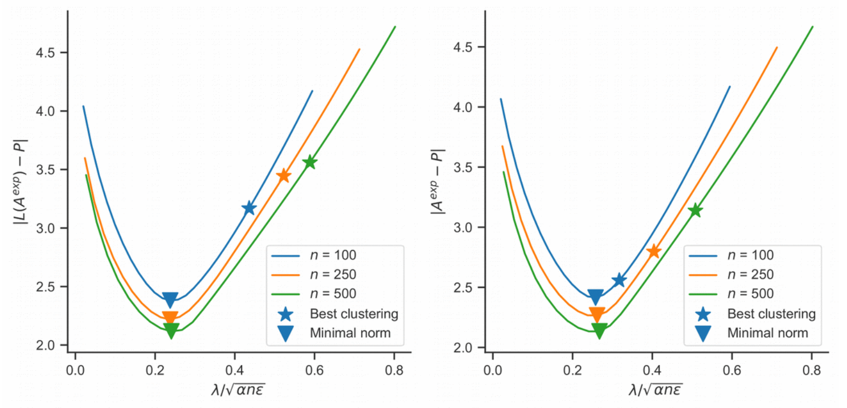


Sketch of sketch of proof

Proof is based on **spectral norm concentration**

$$\|A - \mathbb{E}A\| \qquad \|L(A) - L(\mathbb{E}A)\|$$

+ *classical "Davis-Kahan"-based perturbation analysis*
+ *almost-optimal k-means*



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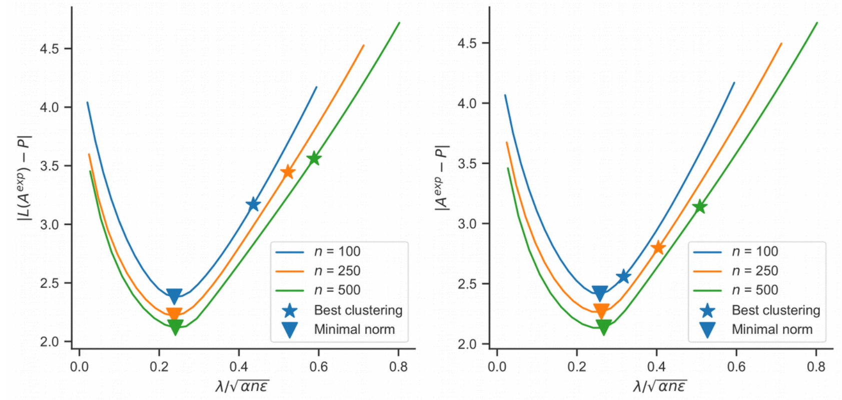

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*Summary (valid for **any matrix with Bernoulli entries**)*

	Static		Dynamic		
	Adjacency	Laplacian	Adjacency	Laplacian	Sparsity
Oliveira 2009	$\log n$	1			$\alpha_n \gtrsim n^{-1} \log n$
Bandeira 2016	$\sqrt{\log n}$				$\alpha_n \gtrsim n^{-1}$
Lei 2015	$\sqrt{\alpha_n n}$				$\alpha_n \gtrsim n^{-1} \log n$
Pensky 2019	$\sqrt{\alpha_n n}$		$\sqrt{\alpha_n n \rho_n}$		$\alpha_n \gtrsim n^{-1} \log n$
Us	$\sqrt{\alpha_n n}$	$\sqrt{1/(\alpha_n n)}$	$\sqrt{\alpha_n n \rho_n}$	$\sqrt{\rho_n/(\alpha_n n)}$	$\alpha_n/\rho_n \gtrsim n^{-1} \log n$

Useful in many other contexts !

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- Choice of forgetting factor in practice?
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Keriven, Vaiter. **Sparse and Smooth: improved guarantees for Spectral Clustering in the Dynamic Stochastic Block Model.** *arXiv:2002.02892*