A short introduction to graphons

Nicolas Keriven

CNRS, Gipsa-lab

Based on the textbook “Large networks and graph limits” (L. Lovasz, 2012)

Graphon : what ?

“ Large graph limits ”

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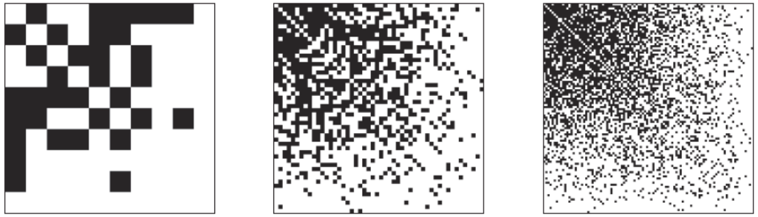
- Notion of **convergence**
  - Which sense ?
  - Towards what ?
  - Which “metric” ?



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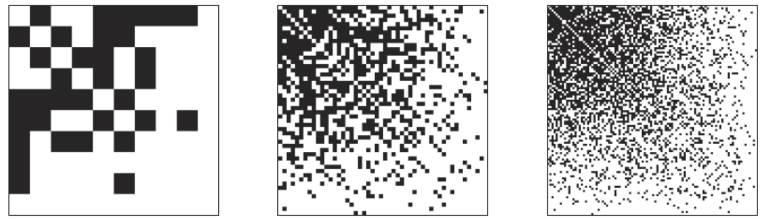

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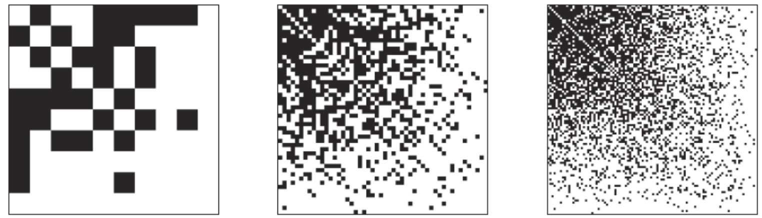

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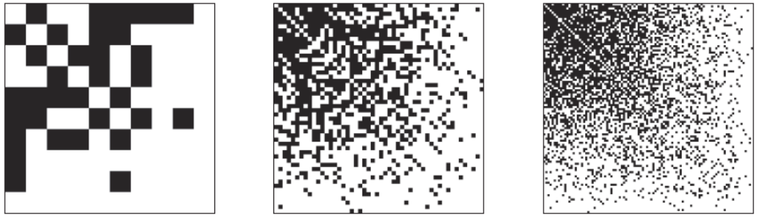

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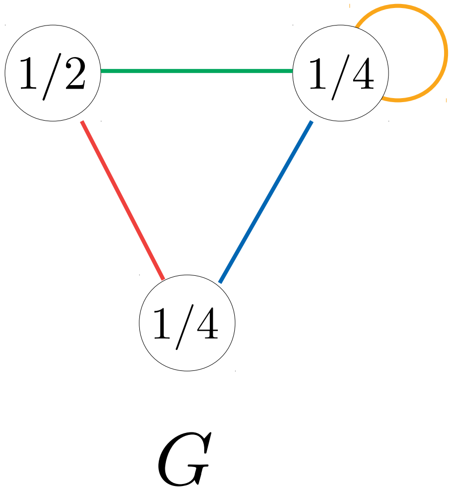

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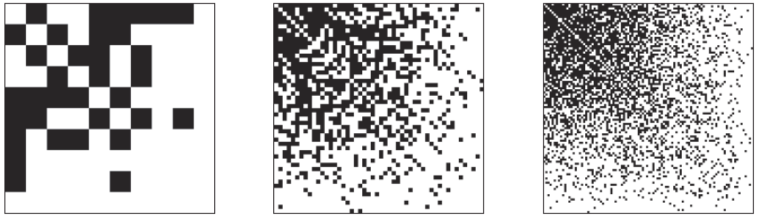

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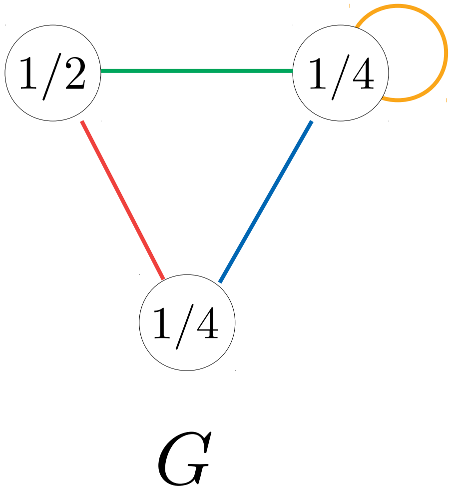

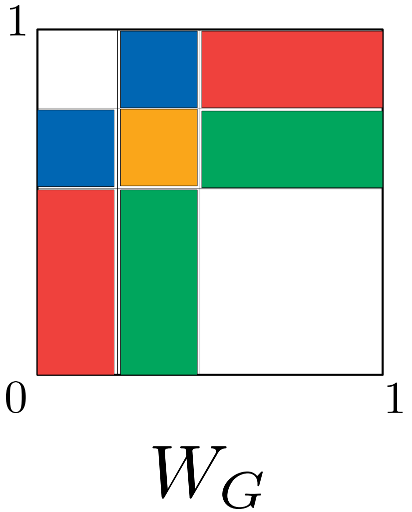

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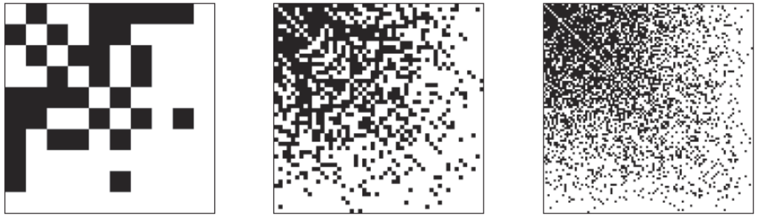

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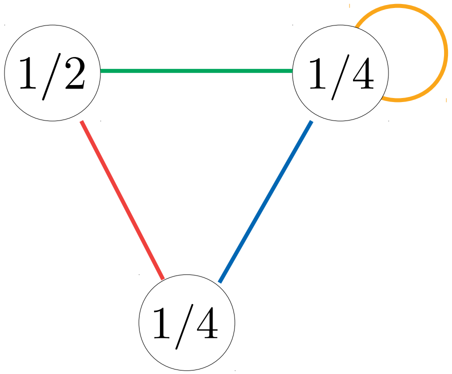

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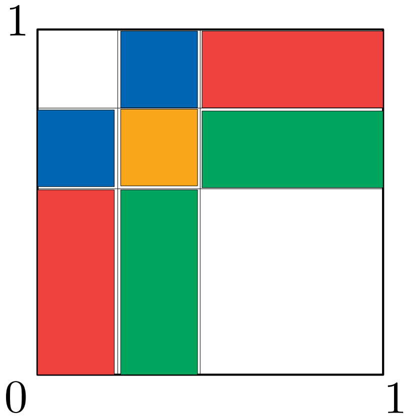

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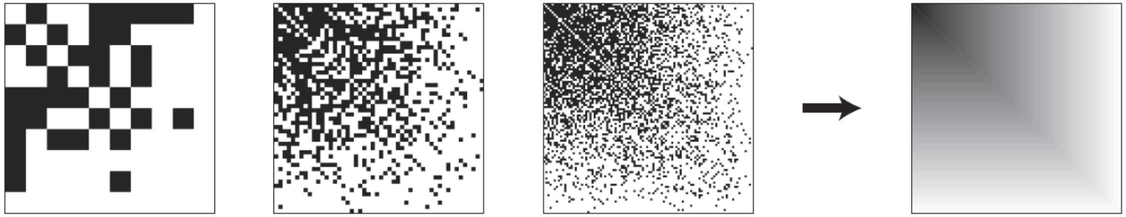

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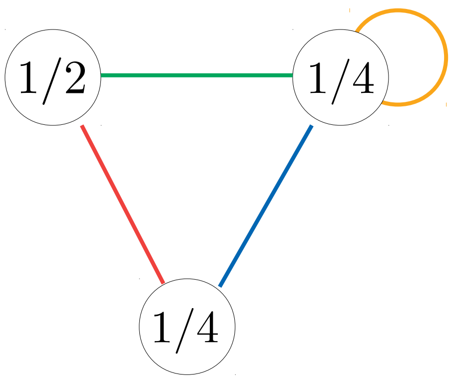

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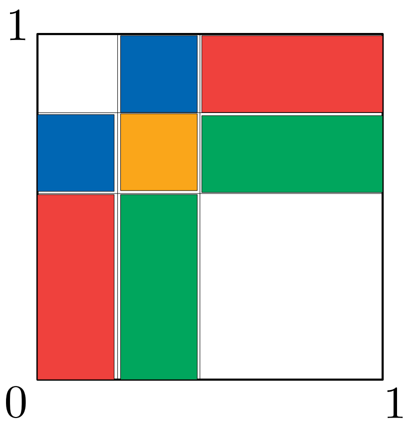

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- *Mostly **theoretical**, but a good generative model for applications*
- *The basic theory is only "satisfying" for **dense graphs***

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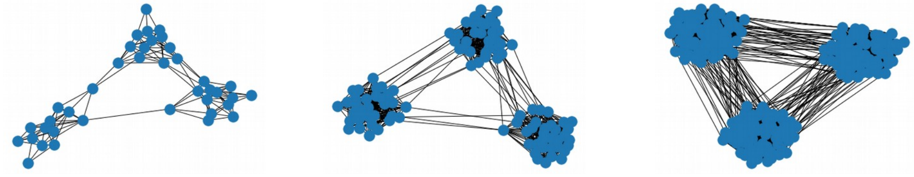

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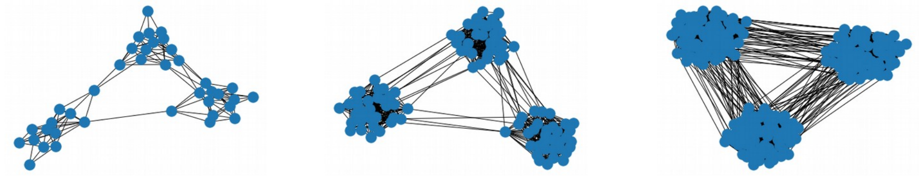

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  - Related to *subgraph sampling*
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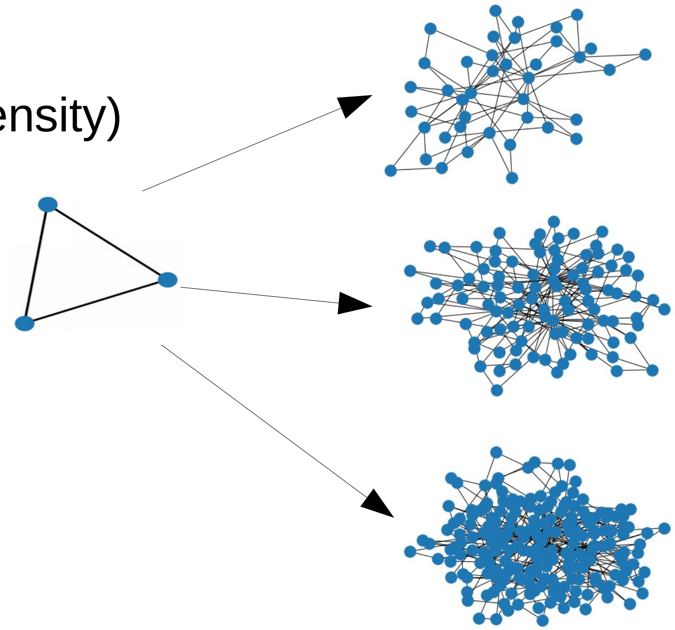

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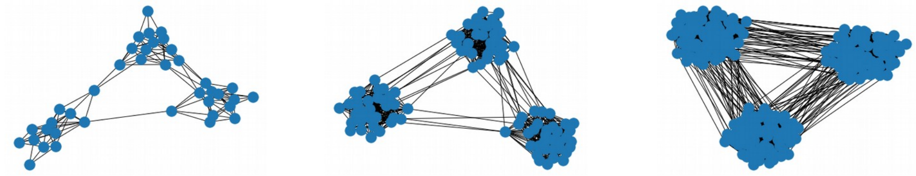

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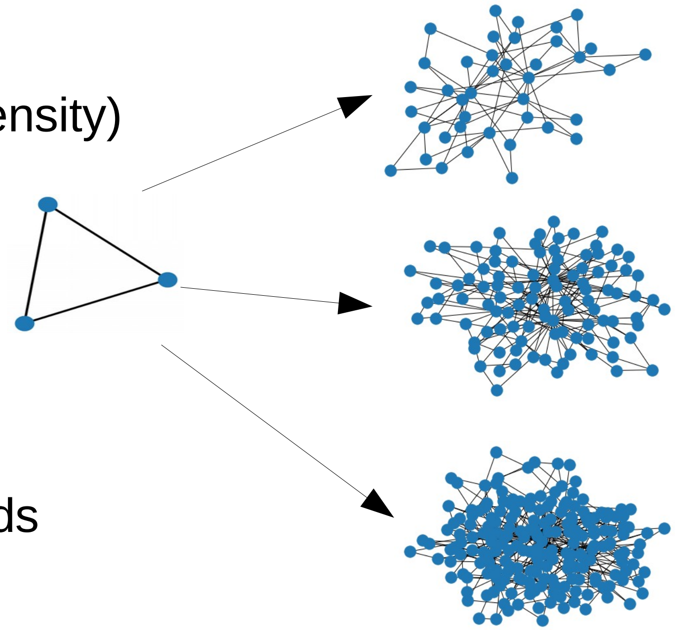

- Convergence of parameters (subgraphs density)
  - Related to *subgraph sampling*
  - Easy to define, more difficult to analyze
- Cut-norm convergence
  - “True” appropriate mathematical notion
  - What really connects several mathematical fields
  - Mathematically “advanced”!

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$W : [0,1]^2 \to [0,1]$ symmetric measurable

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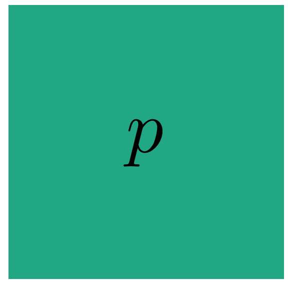

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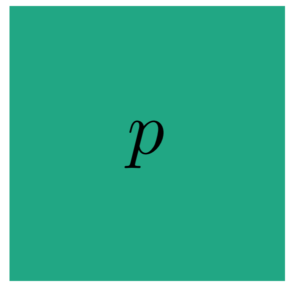

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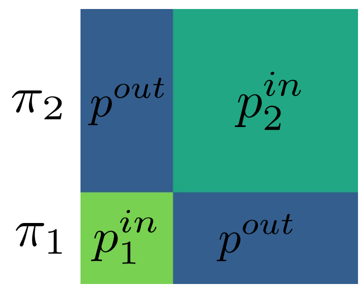

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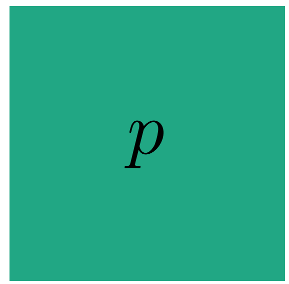

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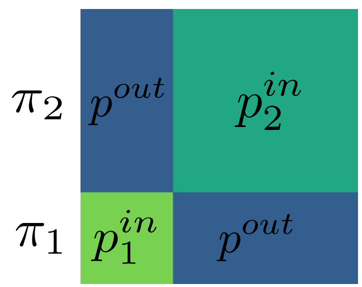

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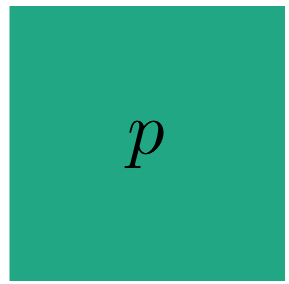

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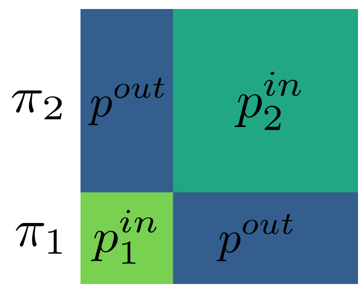

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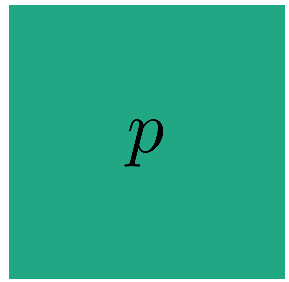

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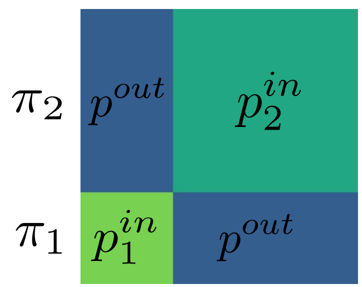

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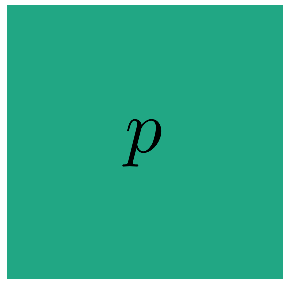

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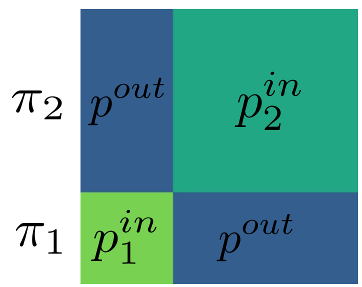

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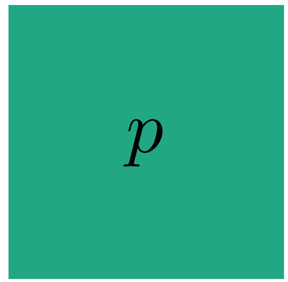

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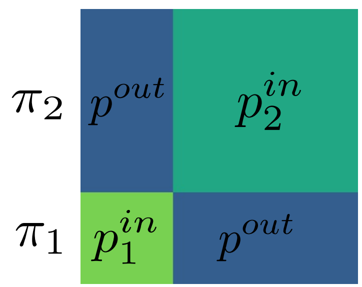

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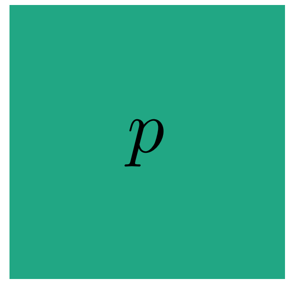

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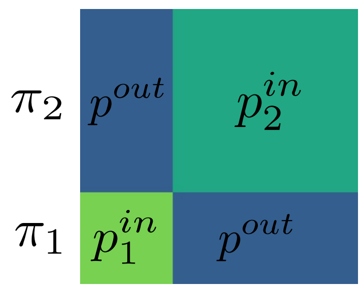

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More interpretable, but often does not change “basic” mathematical properties

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- Hierarchical "exchangeable models" (pertains to invariance by permutation and nesting) *[Bickel and Chen, Veitch, Roy, Orbanz...]*

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Subgraph density: probability of drawing a given subgraph

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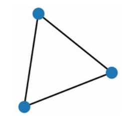
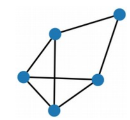
Ex : $t_{ind}($  $,$  $) = 2/10$

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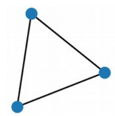
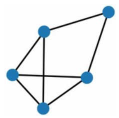
Ex : $t_{ind}($  $,$  $) = 2/10$

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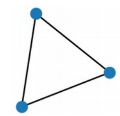
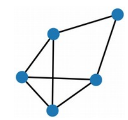
Ex : $t_{ind}($  ,  $) = 2/10$

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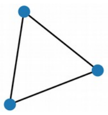
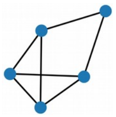
Ex : $t_{ind}($  $,$  $) = 2/10$

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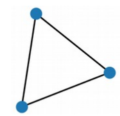
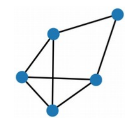
Ex : $t_{ind}($  ,  $) = 2/10$

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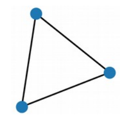
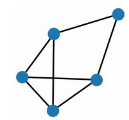
Ex : $t_{ind}($  ,  $) = 2/10$

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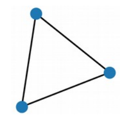
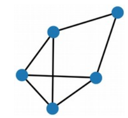
Ex : $t_{ind}($  ,  $) = 2/10$

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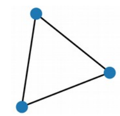
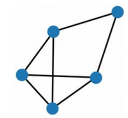
Ex : $t_{ind}($  ,  $) = 2/10$

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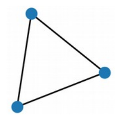
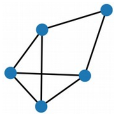
Ex : $t_{ind}($  ,  $) = 2/10$

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Unique up to (weak) ***isomorphism***

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for $\phi : [0, 1] \to [0, 1]$ measure-preserving

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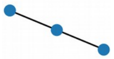
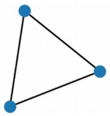
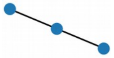
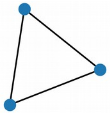
$t_{ind}($ ,  $) = 0$

$t($ ,  $) = 3/27$

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(non-injective, preserves only adjacency)
(chap 5&7: everything equivalent!)

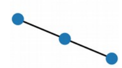
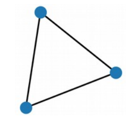
$$t(F,G) = \frac{\hom(F,G)}{n^k}$$

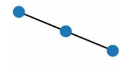
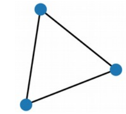
$$t(F,W) = \int_{[0,1]^k} \prod_{ij\in E_F} W(x_i,x_j)dx_1\dots dx_k$$

Somehow simpler: $t(F,G) = t(F,W_G)$

$$|t_{ind}(F,G) - t_{ind}(F,W_G)| \xrightarrow[|G|\to\infty]{} 0$$

Exo 7.7

$t_{ind}($  ,  $) = 0$

$t($  ,  $) = 3/27$

3 : graphon as completed graph space

Cut-norm on kernels (symmetric measurable $U : [0,1]^2 \to \mathbb{R}$)

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Those two theorems really sparked the mathematical interest on graphons. They are **hybrids analysis/combinatoric results**, and have interesting corollaries: eg, for all ε there is n_ε such that graphs of size n_ε are an ε - net for graphons (in the cut metric).

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Inverse counting lemma (lem 10.32)

$$\left[\forall F \in \mathcal{G}_k, |t(F,U) - t(F,W)| \leq 2^{-k^2}\right] \Rightarrow \delta_\square(U,W) \leq \frac{50}{\sqrt{\log k}}$$

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Approximation by step functions: **Szemerédi partitions** regularity Lemma**s** (chap 9)
base of proof for compacity

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Concentration (Lem 10.16)

$$\mathbb{P}(\delta_\square(G_n, W) \geq 22/\sqrt{\log n}) \leq e^{-\frac{n}{2\log n}}$$

What about **sparse** graphon ?

Basic graphon theory only covers **dense** graphs:

if $|E_{G_n}| = o(n^2)$, then $G_n \xrightarrow{t_{ind}} 0$!

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[Veitch, Roy, Orbanz]

Metric point of view

Eg, L^p- graphon *[Borgs et al 2014]*

- Based on some renormalization
- Pbm: dense "spots" converge to infinity

"Graph parameters" point of view

What about **sparse** graphon ?

Basic graphon theory only covers **dense** graphs:

if $|E_{G_n}| = o(n^2)$, then $G_n \xrightarrow{t_{ind}} 0$!

There are many competing theories for sparse graph limits, based on **one of the three points of view**. But no satisfying "triple equivalency"!

Random graphs point of view (most used)

$$a_{ij} \sim \mathrm{Ber}(\rho_n W(x_i, x_j))$$

$\rho_n \sim 1$	Dense
$\rho_n \sim 1/n$	Sparse
$\rho_n \sim \log n/n$	Relatively sparse

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Convergence to ***measures on the square***
[Kunszenti, Lovasz, Szegedy]

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- Not equivalent to anything else !

Thank you !